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On a class of symmetric *CR* manifolds

A. Lotta[a,*] and M. Nacinovich[b]

[a] *Dipartimento di Matematica, Università di Bari, Stanza 20, Piano 3, Via E. Orabona 4, 70125 Bari, Italy*
[b] *Dipartimento di Matematica, Università di Roma "Tor Vergata", Via Della Ricerca Scientifica, 00133 Roma, Italy*

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Abstract

We study and classify a large class of minimal orbits in complex flag manifolds for the holomorphic action of a real Lie group. These orbits are all symmetric *CR* spaces for the restriction of a suitable class of Hermitian invariant metrics on the ambient flag manifold. As a particular case we obtain that the standard compact homogeneous *CR* manifolds associated with semisimple Levi–Tanaka algebras are symmetric *CR*-spaces.

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1. Introduction

The aim of this paper is to study a large class of symmetric *CR* manifolds. These manifolds are a special instance of a very general notion of symmetry recently introduced by Kaup and Zaitsev [6]. Their definition generalizes both Riemannian symmetric and Hermitian symmetric spaces. Examples of symmetric *CR* manifolds derived from symmetric bounded domains were considered in [6]. They can be described as follows. Fix a Hermitian symmetric manifold of the compact type $X = G_u/K$ and let $X_0 = G/K$ be its dual Hermitian symmetric space of the noncompact type. It is known that X can be represented as a complex flag manifold

*Corresponding author. Fax: +39-080-544-2722.
E-mail addresses: lotta@dm.uniba.it (A. Lotta), nacinovi@mat.uniroma2.it (M. Nacinovich).

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$X = G^{\mathbb{C}}/Q$, where as usual $G^{\mathbb{C}}$ denotes the complexification of G, in such a way that X_0 is embedded in X as an *open submanifold*, given by the G-orbit containing the point ϱ corresponding to the coset Q. There exists a unique closed G-orbit M in X; if X_0 is realized as a bounded symmetric domain $D \subset E$ in a suitable finite dimensional complex Hilbert vector space, then M coincides with the Bergman–Shilov boundary of D (cf. [15]). It is proved in [6] that M is a symmetric CR manifold with respect to the restriction of the Hermitian metric of X.

It is natural to ask if this class of examples can be extended to more general minimal orbits in a complex flag manifold $X = G^{\mathbb{C}}/Q$, for the holomorphic action of a real form G of $G^{\mathbb{C}}$. Note that in general X is not Hermitian symmetric (cf. [15]). In this more general context, a systematic study of the CR types of the minimal orbits has been developed in [9], where a complete classification is provided in terms of *minimal parabolic CR algebras* and their corresponding *σ-diagrams*.

In this paper, we study a special class of minimal parabolic CR algebras $(\mathfrak{g}, \mathfrak{q})$, characterized by what we call the (J) *property*. Their corresponding minimal orbits M admit CR symmetries preserving their canonical CR structure as CR submanifolds. They are isometries for special Hermitian metrics obtained by restricting to M suitable invariant Hermitian metrics on the ambient flag manifold X.

We first examine the case where $\mathfrak{g}$ is a *semisimple Levi–Tanaka algebra* (see [8]). A key property of semisimple Levi–Tanaka algebras is that their partial complex structure is induced by an inner derivation [10]. This leads to the existence of CR symmetries on the corresponding minimal orbits (Theorem 5.2). The (J) property is a natural generalization of this property of semisimple Levi–Tanaka algebras (Definition 6.1) and is a sufficient condition for M to be a symmetric CR manifold. In the last part of the paper we obtain a complete classification of the fundamental minimal parabolic CR algebras $(\mathfrak{g}, \mathfrak{q})$ having the (J) property: $\mathfrak{g}$ decomposes into a direct sum of simple ideals $\mathfrak{g}_i$, in such a way that, with respect to the corresponding decomposition of the parabolic subalgebra $\mathfrak{q}$, each $(\mathfrak{g}_i, \mathfrak{q}_i)$ is either a simple Levi–Tanaka algebra of the complex type, or a simple minimal parabolic CR algebras of the real type whose σ-diagram satisfies three additional conditions (A)–(C) (Theorem 8.1). Note that these additional conditions are automatically satisfied when $(\mathfrak{g}_i, \mathfrak{q}_i)$ is Levi–Tanaka.

We give examples of minimal parabolic CR algebras $(\mathfrak{g}, \mathfrak{q})$ which are not of Levi–Tanaka, but have the (J) property. Thus our class of symmetric CR manifolds contains and is larger than the class of standard compact CR manifolds [11].

2. Minimal orbits in complex flag manifolds

A *complex flag manifold* is a compact homogeneous complex manifold of the form $\mathscr{F} = G^{\mathbb{C}}/Q$ where $G^{\mathbb{C}}$ is a connected semisimple complex Lie group and Q a parabolic subgroup of $G^{\mathbb{C}}$. This means that the corresponding Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}^{\mathbb{C}} = Lie(G^{\mathbb{C}})$ is parabolic (i.e. contains a maximal solvable subalgebra of $\mathfrak{g}^{\mathbb{C}}$), and Q

is the normalizer of $\mathfrak{q}$ in $\mathfrak{g}^{\mathbb{C}}$. A real form G of $G^{\mathbb{C}}$ acts in a natural manner on $\mathscr{F}$, and $\mathscr{F}$ decomposes into *finitely many* G-orbits. Among the orbits of G, there is exactly one which is closed. It is compact and is called *minimal*, since it is the G-orbit having the smallest dimension (see for instance [15]).

All orbits of G are *generic CR* submanifolds of $\mathscr{F}$. This means that their type (n, k) is such that $n + k$ equals the complex dimension of $\mathscr{F}$. The real group G acts on M as a transitive group of CR transformations.

We can describe the CR structure of M from a Lie-theoretic point of view by means of the information contained in the pair $(\mathfrak{g}, \mathfrak{q})$: this is called a (finite dimensional) *parabolic CR algebra* in [9]. After replacing, if needed, the parabolic Q by one of its conjugates, we may assume that M is the orbit of the point ϱ of $\mathscr{F}$ corresponding to the coset Q. Let $G_+ := G \cap Q$ be the isotropy subgroup of G relative to the point ϱ, and let $\mathfrak{g}_+ = \mathfrak{g} \cap \mathfrak{q}$ be the Lie algebra of G_+. Set

$$\mathfrak{H}_+ := (\mathfrak{q} + \bar{\mathfrak{q}}) \cap \mathfrak{g} = \{Re\ Z | Z \in \mathfrak{q}\},$$

where conjugation and real parts are taken with respect to the real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. Then, after the natural identification between $T_{\varrho}M$ and $\mathfrak{g}/\mathfrak{g}_+$, the analytic tangent space $H_{\varrho}M$ of M at ϱ is identified with the quotient space

$$\mathfrak{H} := \mathfrak{H}_+ / \mathfrak{g}_+.$$

The partial complex structure J on $H_{\varrho}M$ can be characterized as follows:

$$X + iY \in \mathfrak{q} \iff X, Y \in \mathfrak{H}_+ \quad \text{and} \quad J(X + \mathfrak{g}_+) = Y + \mathfrak{g}_+. \tag{1}$$

The CR structure of M is then completely determined by the requirement that G operates on M as a group of CR diffeomorphisms. In the following we shall restrain our attention to minimal orbits whose CR structure is of *finite type* in the sense of Bloom and Graham. This is equivalent to requiring that $\mathfrak{H}_+$ generates $\mathfrak{g}$ as a Lie algebra. In this case we say that the parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ is *fundamental*.

3. σ-diagrams

One of the main results in [9] is the classification of CR types of *minimal* orbits; the corresponding parabolic CR algebras are also called minimal. The classification is carried out in terms of Satake diagrams with crosses, called σ-diagrams. In this section, we rehearse the relevant information about σ-diagrams.

Fix a parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$; by definition, $\mathfrak{g}$ is a finite-dimensional real semisimple Lie algebra, and $\mathfrak{q}$ is a parabolic subalgebra of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. We shall assume that $\mathfrak{g}_+ := \mathfrak{g} \cap \mathfrak{q}$ does not contain any nonzero ideal of $\mathfrak{g}$; in this case $(\mathfrak{g}, \mathfrak{q})$ is called *transitive*. The integer $k := \dim_{\mathbb{C}} \mathfrak{H}$ is called the *CR dimension* of $(\mathfrak{g}, \mathfrak{q})$, while the difference $s := \dim_{\mathbb{R}}(\mathfrak{g}/\mathfrak{g}_+) - \dim_{\mathbb{R}} \mathfrak{H}$ is called the *CR codimension* of $(\mathfrak{g}, \mathfrak{q})$. If $k = 0$ (resp. $s = 0$), $(\mathfrak{g}, \mathfrak{q})$ is called *totally real* (resp. *totally complex*).

A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called an *adapted* Cartan subalgebra of $(\mathfrak{g}, \mathfrak{q})$ if $\mathfrak{h} \subset \mathfrak{g}_+$. We have as usual

$$\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-,$$

where

$$\mathfrak{h}^- := \{H \in \mathfrak{h} \mid ad_{\mathfrak{g}}(H) \text{ has real eigenvalues}\},$$

$$\mathfrak{h}^+ := \{H \in \mathfrak{h} \mid ad_{\mathfrak{g}}(H) \text{ has purely imaginary eigenvalues}\}.$$

$\mathfrak{h}^-$ and $\mathfrak{h}^+$ are, respectively, the *vectorial part* and the *toroidal part* of $\mathfrak{h}$. The maximal dimension of the vectorial part of a Cartan subalgebra of $\mathfrak{g}$ is called its *real rank*. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called *maximally vectorial* if the dimension of its vectorial part $\mathfrak{h}^-$ equals the real rank.

Having fixed an adapted Cartan subalgebra $\mathfrak{h}$, denote by $\Sigma = \Sigma(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ the root system in $\mathfrak{g}^{\mathbb{C}}$ of its complexification $\mathfrak{h}^{\mathbb{C}}$. The parabolic subalgebra $\mathfrak{q}$ corresponds to a *parabolic set of roots* $\mathcal{Q}$ in $\Sigma(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$. These $\mathcal{Q} \subset \Sigma$ are characterized by the two properties (cf. [4]):

(a) $\mathcal{Q}$ is closed, that is,

$$\forall \alpha, \beta \in \mathcal{Q}, \quad \alpha + \beta \in \Sigma \Rightarrow \alpha + \beta \in \mathcal{Q},$$

(b) $\Sigma = \mathcal{Q} \cup (-\mathcal{Q})$.

The parabolic set $\mathcal{Q}$ is partitioned into its *reductive part* $\mathcal{Q}^r := \{\alpha \in \mathcal{Q} \mid -\alpha \in \mathcal{Q}\}$, and its *nilpotent part* $\mathcal{Q}^n := \mathcal{Q} \backslash \mathcal{Q}^r$ (for more details, see e.g. [9,14]).

Set $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h}^- \oplus i\mathfrak{h}^+$. This is a real form of $\mathfrak{h}^{\mathbb{C}}$, on which the roots of Σ are real valued. The conjugation in $\mathfrak{g}^{\mathbb{C}}$ defines an involution on $\mathfrak{h}_{\mathbb{R}}$. We denote by $\sigma : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$ its dual map: it determines an involution of the root system Σ. Denote by $\bar{\alpha}$ the image of a root $\alpha \in \Sigma$ under σ. We set as usual

$$\Sigma_\bullet := \{\alpha \in \Sigma \mid \bar{\alpha} = -\alpha\}, \quad \Sigma_\circ := \Sigma \backslash \Sigma_\bullet.$$

Fix a Weyl chamber $C \subset \mathfrak{h}^*_{\mathbb{R}}$ of Σ, and let $\Delta(C)$ and $\Sigma^+(C)$ be, respectively, the corresponding sets of fundamental and of positive roots. Since the Weyl group $W(\Sigma)$ acts simply transitively on the set of Weyl chambers, there is a unique element $w_C \in W(\Sigma)$ such that

$$w_C(\bar{C}) = C.$$

Denote by ε_C the composition:

$$\varepsilon_C := w_C \circ \sigma.$$

We say that $\Delta(C)$ is a *σ-fundamental* system of roots (cf. [14]) if

$$\text{for all } \alpha \in \Sigma^+ \setminus \Sigma_\bullet, \quad \bar{\alpha} \in \Sigma^+(C).$$

In this case the conjugation σ is expressed by

$$\bar{\alpha} = \varepsilon_C(\alpha) + \sum_{\beta \in \Delta_\bullet(C)} n_{\bar{\alpha},\beta}\beta \text{ with } n_{\bar{\alpha},\beta} \in \mathbb{Z},\ n_{\bar{\alpha},\beta} \geqslant 0, \quad \forall \alpha \in \Delta(C) \setminus \Sigma_\bullet,$$

where we have set $\Delta_\bullet(C) \coloneqq \Delta(C) \cap \Sigma_\bullet$.

With this notation, the minimal transitive parabolic *CR* algebras can be characterized by the following properties:

(1) There exists a maximally vectorial adapted Cartan subalgebra $\mathfrak{h}$ of $(\mathfrak{g}, \mathfrak{q})$.
(2) The set $\mathcal{Q} \subset \Sigma(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ of roots associated to $\mathfrak{q}$ contains a σ-fundamental system Δ'.

Hence the *σ-diagram* of a minimal parabolic *CR* algebra is defined as follows. Choose an adapted maximally vectorial Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a σ-fundamental system of roots Δ such that $\Sigma^- \subset \mathcal{Q}$. Then the σ-diagram of $(\mathfrak{g}, \mathfrak{q})$ is the Satake diagram with crosses which is obtained from the Satake diagram of $\mathfrak{g}$ (see [13,14]) by adding a cross on those circles which correspond to the roots in $\Delta_\times \coloneqq \Delta \setminus \mathcal{Q}$.

It is proved in [9] that there is a one to one correspondence between classes of *CR* isomorphic minimal parabolic *CR* algebras and σ-diagrams.

The fundamental, transitive minimal parabolic *CR* algebras are characterized by the following property of their σ-diagram:

Theorem 3.1 (Medori and Nacinovich [9]). *Let $(\mathfrak{g}, \mathfrak{q})$ be a transitive minimal parabolic CR algebra. Fix an adapted maximally vectorial Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a σ-fundamental system of roots $\Delta \subset \Sigma(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ such that $\Sigma^- \subset \mathcal{Q}$. Set $\Delta_\circ \coloneqq \Delta \cap \Sigma_\circ$ and $\Delta_{\overset{\circ}{\times}} \coloneqq \Delta_\circ \cap \Delta_\times$. Then the following properties are equivalent*:

(a) $(\mathfrak{g}, \mathfrak{q})$ *is fundamental*;
(b) *If* $\alpha \in \Delta_{\overset{\circ}{\times}}$, *then* $\varepsilon_C(\alpha) \notin \Delta_\times$.

4. Symmetric *CR* manifolds

An almost *CR* manifold is a triple (M, HM, J) consisting of a smooth paracompact real manifold M, a smooth subbundle HM of its tangent bundle TM, and an anti-involution $J : HM \to HM$ defining a complex structure on each fiber of HM.

Let $\mathcal{T}^{1,0}(M) = \{X + iJX \mid X \in \Gamma HM\}$. The partial complex structure J is *formally integrable* if $[\mathcal{T}^{1,0}(M), \mathcal{T}^{1,0}(M)] \subset \mathcal{T}^{1,0}(M)$. If this condition is satisfied we say that (M, HM, J) is a *CR* manifold.

Let $\mathscr{D}_\infty M$ the Lie algebra of vector fields on M generated by the smooth sections of HM. If $(\mathscr{D}_\infty M)_x = T_x M$ for all $x \in M$, we say that M is *of finite type*.

A partial Hermitian structure on (M, HM, J) is a Riemannian metric g on M such that, for all sections $X, Y \in \Gamma HM$ we have

$$g(JX, JY) = g(X, Y).$$

A smooth map $\phi : M \to M$ is *CR* if $d\phi(HM) \subset HM$ and $J \circ d\phi = d\phi \circ J$ on HM.

Definition 4.1 (Kaup and Zaitsev [6]). We say that (M, HM, J, g) is a *symmetric CR manifold* if for each point $x \in M$ there exists a *CR* diffeomorphism $\sigma_x : M \to M$ satisfying the following requirements:

(1) σ_x is an isometry with respect to the Riemannian metric g;
(2) $\sigma_x(x) = x$;
(3) The restriction to $H_x M \oplus (\mathscr{D}_\infty M)_x^\perp$ of the differential $(\sigma_x)_*$ of σ_x at x equals $-Id$.

The *CR*-diffeomorphism σ_x is called the *symmetry* of M at the point x. It is uniquely determined by the above conditions (1)–(3) and is an involutive transformation of M (see [6, Proposition 3.2]). Every symmetric almost *CR* manifold is *CR-homogeneous*: the group of all isometric *CR* diffeomorphisms of M is a Lie group acting transitively on M, with compact isotropy subgroups.

5. Symmetric *CR* manifolds associated to semisimple Levi–Tanaka algebras

To each semisimple Levi–Tanaka algebra we can associate a minimal orbit M in a complex flag manifold. In this section, we show that this M is a symmetric *CR* manifold for the restriction of a suitable invariant Hermitian metric of the ambient flag manifold. Let $\mathfrak{g} = \bigoplus_{p \geqslant -\mu} \mathfrak{g}_p$ be a $\mathbb{Z}$-graded real Lie algebra. A partial complex structure on $\mathfrak{g}$ is a complex structure $J : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ which satisfies

$$[JX, JY] = [X, Y] \quad \forall X, Y \in \mathfrak{g}_{-1},$$

$$[A, JX] = J[A, X] \quad \forall A \in \mathfrak{g}_0, \ \forall X \in \mathfrak{g}_{-1}.$$

We recall that $\mathfrak{g} = \bigoplus_{p \geqslant -\mu} \mathfrak{g}_p$ is called *transitive* if $[X, \mathfrak{g}_{-1}] \neq 0$ for $X \in \bigoplus_{p \geqslant 0} \mathfrak{g}_p \setminus \{0\}$; *fundamental* if the Lie subalgebra $\mathfrak{m} = \bigoplus_{-\mu \leqslant p < 0} \mathfrak{g}_p$ is generated by $\mathfrak{g}_{-1}$.

The pair $(\mathfrak{g}, J)$ consisting of a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{p \geqslant -\mu} \mathfrak{g}_p$ and of a partial complex structure J on $\mathfrak{g}$ is called a Levi–Tanaka algebra if:

(1) $\mathfrak{g}$ is fundamental, i.e. $\mathfrak{g}_{-1}$ generates the Lie subalgebra $\mathfrak{m} := \bigoplus_{p<0} \mathfrak{g}_p$ of $\mathfrak{g}$;
(2) The adjoint representation gives an isomorphism between $\mathfrak{g}_0$ and the algebra of 0-degree derivations of $\mathfrak{m}$ whose restriction to $\mathfrak{g}_{-1}$ commutes with J;
(3) $\mathfrak{g}$ is the maximal transitive prolongation of the graded Lie algebra $\mathfrak{m} \oplus \mathfrak{g}_0$.

For basic facts about Levi–Tanaka algebras, we refer the reader to [8,10]. Here we shall be interested to the semisimple case. If $\mathfrak{g} = \oplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ is semisimple, then $(\mathfrak{g}, J)$ is a Levi–Tanaka algebra provided that no nontrivial simple ideal of $\mathfrak{g}$ is contained in $\oplus_{p=-1}^{\mu} \mathfrak{g}_p$. A very important feature of semisimple Levi–Tanaka algebras is that the partial complex structure J extends to a zero degree inner derivation, and hence there exists a unique vector in the center of $\mathfrak{g}_0$, that we still denote by J, such that

$$[J, X] = JX \quad \forall X \in \mathfrak{g}_{-1}.$$

To a semisimple Levi–Tanaka algebra $\mathfrak{g} = \oplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ we associate a minimal parabolic transitive CR algebra $(\mathfrak{g}, \mathfrak{q})$ as follows. Let $\mathfrak{g}_{-1}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}_{-1}$; the complex structure $J : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ extends to a $\mathbb{C}$-linear operator on $\mathfrak{g}_{-1}^{\mathbb{C}}$, denoted by the same symbol, such that $J^2 = -Id$. Accordingly, we have a direct sum decomposition:

$$\mathfrak{g}_{-1}^{\mathbb{C}} = \mathfrak{m}^{01} \oplus \mathfrak{m}^{10},$$

where $\mathfrak{m}^{01} = \{X + iJX \mid X \in \mathfrak{g}_{-1}\}$ and $\mathfrak{m}^{10} = \{X - iJX \mid X \in \mathfrak{g}_{-1}\}$ are the eigenspaces of J, relative to the eigenvalues $-i$ and i, respectively. Notice that, for the conjugation in $\mathfrak{g}^{\mathbb{C}}$ associated to the real form $\mathfrak{g}$, we have:

$$\overline{\mathfrak{m}^{01}} = \mathfrak{m}^{10}.$$

Set

$$\mathfrak{q} := \mathfrak{m}^{01} \oplus \mathfrak{g}_{+}^{\mathbb{C}}, \quad \text{where } \mathfrak{g}_{+} := \bigoplus_{p \geqslant 0} \mathfrak{g}_p.$$

It is proved in [8, Lemma 4.12], that $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Hence $(\mathfrak{g}, \mathfrak{q})$ is a parabolic CR algebra. Actually, $(\mathfrak{g}, \mathfrak{q})$ is minimal and fundamental.

The corresponding minimal orbit can be described as follows. The simply connected homogeneous manifold $S(\mathfrak{g}) = G(\mathfrak{g})/G_{+}(\mathfrak{g})$, where $G(\mathfrak{g})$ is the simply connected Lie group with Lie algebra $\mathfrak{g}$, and $G_{+}(\mathfrak{g})$ is the (closed) analytic subgroup corresponding to $\mathfrak{g}_{+}$, is a $G(\mathfrak{g})$-homogeneous CR manifold in a natural way (see [8]). $S(\mathfrak{g})$ is called the standard homogeneous CR manifold associated to $\mathfrak{g}$. Consider the complex flag manifold $\mathscr{F} = G^{\mathbb{C}}/Q$, where $G^{\mathbb{C}}$ is the simply connected complex Lie group corresponding to $\mathfrak{g}^{\mathbb{C}}$, and $Q \subset G^{\mathbb{C}}$ is the normalizer of $\mathfrak{q}$. Let $G \subset G^{\mathbb{C}}$ be the analytic subgroup of $G^{\mathbb{C}}$ corresponding to $\mathfrak{g}$ and M the G-orbit of the point $\underline{o} \in \mathscr{F}$ corresponding to the coset Q. It is showed in [8, Theorem 4.9], that there is a canonical CR-immersion of the standard homogeneous CR manifold $S(\mathfrak{g})$ into $\mathscr{F}$, whose image is the orbit M. This map is actually a CR diffeomorphism (see [12]). Since $S(\mathfrak{g})$ is compact (see [11]), M is the minimal orbit corresponding to $(\mathfrak{g}, \mathfrak{q})$ and is a CR manifold of finite type.

Let $\mathfrak{h} \subset \mathfrak{g}_0$ be an adapted maximally vectorial Cartan subalgebra of $(\mathfrak{g}, \mathfrak{q})$, and $\Sigma = \Sigma(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}) \subset \mathfrak{h}_{\mathbb{R}}^{*}$ the corresponding root system. Fix a Weyl chamber C such that

$\Delta = \Delta(C)$ is σ-fundamental and $\Sigma^- \subset \mathcal{Q}$ (notation as in Section 3). We also fix an *adapted* Cartan decomposition of $\mathfrak{g}$ according to [11]:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

and we have

$$\mathfrak{k} = \bigoplus_{-\mu}^{0} \mathfrak{k}_{|j|}, \quad \mathfrak{p} = \bigoplus_{-\mu}^{0} \mathfrak{p}_{|j|},$$

with $\mathfrak{k}_{|0|} = \mathfrak{k} \cap \mathfrak{g}_0$, $\mathfrak{k}_{|j|} = \mathfrak{k} \cap (\mathfrak{g}_j \oplus \mathfrak{g}_{-j})$ for $j < 0$, and similarly for $\mathfrak{p}$. The reductive subalgebra $\mathfrak{g}_0$ inherits a Cartan decomposition given by

$$\mathfrak{g}_0 = \mathfrak{k}_{|0|} \oplus \mathfrak{p}_{|0|}.$$

Note that $J \in \mathfrak{k}_{|0|}$. Let $\mathfrak{g}_u$ be a compact real form of $\mathfrak{g}^{\mathbb{C}}$ such that

$$\mathfrak{k} = \mathfrak{g} \cap \mathfrak{g}_u, \quad \mathfrak{p} = \mathfrak{g} \cap (i\mathfrak{g}_u).$$

The corresponding compact real form G_u of the group $G^{\mathbb{C}}$ acts transitively on $\mathscr{F}$ (see [15]); hence, as a real manifold, $\mathscr{F}$ can be identified with the homogeneous space $G_u/(G_u \cap Q)$. We have: $Lie(G_u \cap Q) = \mathfrak{g}_u \cap \mathfrak{q}^r$. Let $\{H_\alpha, X_\alpha, X_{-\alpha}\}$ be a Weyl basis for $\mathfrak{g}^{\mathbb{C}}$, such that

$$\mathfrak{g}_u = \sum_{\alpha \in \Sigma} \mathbb{R}(iH_\alpha) \oplus \sum_{\alpha \in \Sigma} \mathbb{R}(X_\alpha - X_{-\alpha}) \oplus \sum_{\alpha \in \Sigma} \mathbb{R}i(X_\alpha + X_{-\alpha}).$$

The homogeneous space $G_u/(G_u \cap Q)$ is *reductive* in the sense of [7, Chapter X]. Indeed we have a direct sum decomposition:

$$\mathfrak{g}_u = (\mathfrak{g}_u \cap \mathfrak{q}^r) \oplus V,$$

with an $Ad(G_u \cap Q)$-invariant vector subspace V defined by

$$V := \sum_{\alpha \in \Sigma \setminus \mathcal{Q}^r} (\mathbb{R}i(X_\alpha + X_{-\alpha}) \oplus \mathbb{R}(X_\alpha - X_{-\alpha})).$$

Let $C_{\mathcal{Q}}$ be the subset of the closure $\bar{C}$ in $\mathfrak{h}_{\mathbb{R}}^*$ of the Weyl chamber C, consisting of all vectors λ satisfying:

$$\langle \alpha, \lambda \rangle = 0 \quad \forall \alpha \in \mathcal{Q}^r, \quad \langle \alpha, \lambda \rangle < 0 \text{ if } \alpha \in \mathcal{Q}^n.$$

Here $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathfrak{h}_{\mathbb{R}}^*$ dual of the one induced on $\mathfrak{h}_{\mathbb{R}}$ by the Killing form. The formula shows that the definition of $C_{\mathcal{Q}}$ is independent of the choice of the *good* Weyl chamber C.

The (real) tangent space $T_{\underset{\sim}{o}}\mathscr{F}$ can be canonically identified with V. Its complexification $T^{\mathbb{C}}_{\underset{\sim}{o}}\mathscr{F}$ is given by

$$T^{\mathbb{C}}_{\underset{\sim}{o}}\mathscr{F} = \sum_{\alpha\in\mathscr{Q}^n} \mathfrak{g}^{\alpha} \oplus \sum_{\alpha\in\mathscr{Q}^n} \mathfrak{g}^{-\alpha} = T^{1,0}_{\underset{\sim}{o}}\mathscr{F} \oplus T^{0,1}_{\underset{\sim}{o}}\mathscr{F}.$$

The set $\{X_\alpha, \bar{X}_\alpha\}_{\alpha\in\mathscr{Q}^n}$ is a basis of the complex vector space $T^{\mathbb{C}}_{\underset{\sim}{o}}\mathscr{F}$; let $\{\xi^\alpha, \bar{\xi}^\alpha\}_{\alpha\in\mathscr{Q}^n}$ denote its dual basis in $(T^{\mathbb{C}}_{\underset{\sim}{o}}\mathscr{F})^*$. We have the following characterization of the G_u-invariant Hermitian metrics on $\mathscr{F}$ (cf. [1,3]):

Theorem 5.1. *The G_u-invariant Hermitian metrics on $\mathscr{F}$ are exactly those whose determination at the point $\underset{\sim}{o}$ is given by*

$$h = \sum_{-\alpha\in\mathscr{Q}^n} c_\alpha \xi^\alpha \otimes \bar{\xi}^\alpha \tag{2}$$

where, for each $\alpha\in\mathscr{Q}^n$, c_α is a positive real number. Hence the set of G_u-invariant Hermitian metrics on $\mathscr{F}$ is parametrized by $(\mathbb{R}^+)^n$, $n = \dim_{\mathbb{C}}\mathscr{F}$.

Moreover, metric (2) *is Kähler if and only if*

$$c_\alpha = \langle \lambda, \alpha \rangle$$

for a vector λ in $C_{\mathscr{Q}}$. Hence the G_u-invariant Kähler metrics on $\mathscr{F}$ are parametrized by the points of $C_{\mathscr{Q}} \subset \bar{C}$.

Now we prove the following:

Theorem 5.2. *Let $(\mathfrak{g}, J)$ be a semisimple Levi–Tanaka algebra, with corresponding minimal parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$. Let M be the minimal orbit $G\cdot\underset{\sim}{o}$ in the complex flag manifold $\mathscr{F} = G^{\mathbb{C}}/Q = G_u/G_u\cap Q$. Let h be any G_u-invariant hermitian metric on $\mathscr{F}$. Then M is a symmetric CR manifold with respect to its canonical structure of CR submanifold of $\mathscr{F}$ and the Hermitian metric g on M induced by h.*

Proof. The analytic subgroup K of G, corresponding to $\mathfrak{k}$, acts transitively on the closed orbit M. The isotropy subgroup at $\underset{\sim}{o}$ is $K\cap Q$ (cf. [15, Corollary 3.4]). Since $K\subset G_u\cap G$, the elements of K act at the same time as CR diffeomorphisms and as isometries for the Riemannian metric g on M.

The element $J\in\mathfrak{g}_0$ that induces the partial complex structure on $\mathfrak{g}_{-1}$ belongs to $\mathfrak{k}_{|0|} = \mathfrak{k}\cap\mathfrak{g}_0\subset\mathfrak{k}\cap\mathfrak{q}$. Then $a := \exp(\pi J)$ is an element of $K\cap Q$. As a CR transformation of M, it is a CR symmetry, according to Definition 4.1. Indeed, under the natural identification $T_{\underset{\sim}{o}}M \cong \mathfrak{m}$, its differential a_* at $\underset{\sim}{o}$ is given by $Ad(a) : \mathfrak{m}\to\mathfrak{m}$. Since $ad(J)^2 = -Id$ on $\mathfrak{g}_{-1}$, it follows that $Ad(a) = e^{ad(\pi J)}$ equals $-Id$ on $\mathfrak{g}_{-1}$. Equivalently, a_* equals $-Id$ on $H_{\underset{\sim}{o}}M \cong \mathfrak{g}_{-1}$. Since the metric h is K-invariant and M is K-homogeneous, we deduce the existence of the CR symmetry at all points of M. $\square$

Remark 5.3. Note that, in general, the complex flag manifold $\mathscr{F}$ in which M is embedded is *not* a Hermitian symmetric space. Indeed, according to [15, Lemma 9.22], this is the case if and only if $\mathfrak{q}^n$ is *Abelian*.

For example, when the Levi–Tanaka algebra $(\mathfrak{g}, J)$ is simple of the real type A_ℓ (in this case $\mathfrak{g} \simeq \mathfrak{su}(p, q)$ with $p \cdot q \geqslant 2, p + q = \ell + 1$), then $\mathfrak{q}^n$ is Abelian if and only if the σ-diagram of $(\mathfrak{g}, \mathfrak{q})$ contains only one cross, i.e. if and only if $\mathfrak{q}$ is *maximal* parabolic: thus the complex flag manifold $\mathscr{F}$ is the Grassmannian of k-dimensional vector subspaces of $\mathbb{C}^{\ell+1}$ for some $1 \leqslant k \leqslant \ell$.

6. The (J) property for minimal parabolic CR algebras

Let $(\mathfrak{g}, \mathfrak{q})$ be a transitive minimal parabolic CR algebra. We keep the notation of Sections 2 and 3. In particular we set $\mathfrak{g}_+ = \mathfrak{g} \cap \mathfrak{q}$, $\mathfrak{H}_+ = (\mathfrak{q} + \bar{\mathfrak{q}}) \cap \mathfrak{g}$ and $\mathfrak{H} = \mathfrak{H}_+ / \mathfrak{g}_+$. Since $\mathfrak{g}_+$ is a Lie subalgebra of $\mathfrak{g}$ with $[\mathfrak{g}_+, \mathfrak{H}_+] \subset \mathfrak{H}_+$, the adjoint representation defines a representation

$$\rho : \mathfrak{g}_+ \to \mathfrak{gl}(\mathfrak{H}).$$

By the definition of the complex structure $J \in \mathfrak{gl}(\mathfrak{H})$:

$$J(X + \mathfrak{g}_+) = Y + \mathfrak{g}_+ \iff X, Y \in \mathfrak{H}_+ \quad \text{and} \quad X + iY \in \mathfrak{q}, \tag{3}$$

the maps $\rho(A)$ are $\mathbb{C}$-linear, i.e. commute with J, on $\mathfrak{H}$.

Since $\mathfrak{g}$ is semisimple, the adjoint representation $ad_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is faithful. We shall say that an element $X \in \mathfrak{g}$ is semisimple (resp. nilpotent) iff $ad_{\mathfrak{g}}(X)$ is semisimple (resp. nilpotent) as an element of $\mathfrak{gl}(\mathfrak{g})$. For each $X \in \mathfrak{g}$ there is a unique Wedderburn decomposition $X = X_s + X_n$ where X_s is semisimple, X_n is nilpotent, and $[X_s, X_n] = 0$.

Definition 6.1. We say that a minimal transitive parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ has the *(J) property* if there exists a vector $Z \in \mathfrak{g}_+$ such that $\rho(Z) = J$.

Lemma 6.2. *Assume that the transitive minimal parabolic CR algebra* $(\mathfrak{g}, \mathfrak{q})$ *has the* (J) *property. Then there exists a semisimple element* $\tilde{J}$, *belonging to the radical of* $\mathfrak{g}_+$, *such that*

(i) *all eigenvalues of* $ad_{\mathfrak{g}}(\tilde{J})$ *are purely imaginary*;
(ii) $\rho(\tilde{J}) = J$.

Proof. Let $Z \in \mathfrak{g}_+$ be such that $\rho(Z) = J$. If $Z = Z_s + Z_n$ is the Wedderburn decomposition of Z, then $\rho(Z_s) = J$, because J is semisimple. Let $\mathscr{S}$ denote the set of semisimple elements of $\mathfrak{g}$ and $\mathscr{T}$ the set of all Abelian subalgebras of $\mathfrak{g}$ contained in $\mathscr{S}$. Since $\mathfrak{g}$ (identified with a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ via the adjoint representation) is splittable, according to [4, Chapter VII, Section 5, Proposition 5], a maximal element $\mathfrak{h}$ of $\mathscr{T}$, containing Z_s and contained in $\mathfrak{g}_+$ is a Cartan subalgebra of $\mathfrak{g}$.

The Cartan subalgebra $\mathfrak{h}$ decomposes into the direct sum $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$ of its toroidal and its vectorial part. Accordingly, we have $Z_s = J' + Z_s'$, with $J' \in \mathfrak{h}^+$ and $Z_s' \in \mathfrak{h}^-$. Since the eigenvalues of J are $\pm i$, we obtain $J = \rho(J')$.

Next we consider the decomposition of $\mathfrak{g}_+$:

$$\mathfrak{g}_+ = \mathfrak{n}_+ \oplus \mathfrak{z}_+ \oplus \mathfrak{s},$$

where $\mathfrak{n}_+$ is the nilpotent ideal of $\mathfrak{g}_+$ consisting of all nilpotent elements of its radical, and $\mathfrak{z}_+ \oplus \mathfrak{s}$ is a reductive complement of $\mathfrak{n}_+$ that contains the Cartan subalgebra $\mathfrak{h}$, with center $\mathfrak{z}_+$ and semisimple ideal $\mathfrak{s}$. Let $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_t$ be the decomposition of $\mathfrak{s}$ into the direct sum of its simple ideals.

We also consider the decomposition

$$\mathfrak{H}_+ = \mathfrak{g}_+ \oplus V$$

into the direct sum of two $(\mathfrak{z}_+ \oplus \mathfrak{s})$-invariant subspaces.

We have a unique decomposition of J' into:

$$J' = J_0' + J_1' + \cdots + J_t', \quad \text{with} \quad J_0' \in \mathfrak{z}_+ \quad \text{and} \quad J_i' \in \mathfrak{s}_i \quad \text{for} \quad i = 1, \dots, t.$$

Let $r_i : \mathfrak{s}_i \to \mathfrak{gl}(V)$, for $i = 1, \dots, t$ denote the representation induced by the adjoint representation. Since $\mathfrak{s}_i$ is simple, r_i is either faithful or trivial. We canonically identify V with $\mathfrak{H}$. Then r_i is the restriction to $\mathfrak{s}_i$ of the $\mathbb{C}$-linear representation ρ. Thus, from $\rho([J', A]) = [J, \rho(A)] = 0$ for all $A \in \mathfrak{g}_+$, we obtain that, for all $A \in \mathfrak{s}_i$, we have $r_i([J_i', A]) = 0$ because $[J_i', A] = [J', A]$. This yields $[J_i', V] = 0$ and therefore $\tilde{J} = J_0'$ satisfies all requirements of the lemma. The proof is complete. $\square$

Corollary 6.3. *Let $(\mathfrak{g}, \mathfrak{q})$ be a fundamental, transitive minimal parabolic CR algebra. If $(\mathfrak{g}, \mathfrak{q})$ has the (J) property then there exists a maximally vectorial Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}_+$ and containing an element $\tilde{J}$ of the radical of $\mathfrak{g}_+$ such that $\rho(\tilde{J}) = J$, i.e.*

$$X + i[\tilde{J}, X] \in \mathfrak{q} \quad \forall X \in \mathfrak{H}_+.$$

Proof. Indeed the element $\tilde{J}$ found in the previous lemma belongs to the radical $\mathfrak{r}_+$ of $\mathfrak{g}_+$ and hence to a maximal Abelian subalgebra of $\mathfrak{r}_+$ consisting of semisimple elements. All such algebras are conjugate by an inner automorphism of $\mathfrak{r}_+$. This shows that every Cartan subalgebra (and in particular a maximal vectorial one) contains a conjugate of $\tilde{J}$. This in turn implies that $\tilde{J}$ belongs to a maximal vectorial Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}_+$. $\square$

Every semisimple Levi–Tanaka algebra $(\mathfrak{g}, J)$ has the (J) property: indeed we already noticed that there exists $\tilde{J} \in \mathfrak{g}_0$ such that $[\tilde{J}, X] = JX$ for all $X \in \mathfrak{g}_{-1}$.

It suffices then to observe that $\mathfrak{H} \simeq \mathfrak{g}_{-1}$ and that $\rho(\tilde{J})$ is the restriction to $\mathfrak{g}_{-1}$ of $ad_{\mathfrak{g}}(\tilde{J})$.

The geometric significance of the (J) property is given by the following result which generalizes Theorem 5.2:

Theorem 6.4. *Let* $(\mathfrak{g}, \mathfrak{q})$ *be a fundamental, transitive minimal parabolic CR algebra. Let* $M = G \cdot \underline{o}$ *be the corresponding minimal orbit in the flag manifold* $\mathscr{F} = G^{\mathbb{C}}/Q$.

If $(\mathfrak{g}, \mathfrak{q})$ *has the* (J) *property then there exists a compact real form* G_u *of* $G^{\mathbb{C}}$, *such that* M *is a symmetric CR manifold for the restriction of any* G_u*-invariant Hermitian metric of the ambient complex flag manifold* $\mathscr{F}$.

Proof. We use the notation of the previous lemma: in particular $\mathfrak{h} \subset \mathfrak{g}_+$ is a Cartan subalgebra of $\mathfrak{g}$ whose toroidal part $\mathfrak{h}^+$ contains an element $\tilde{J}$ with $\rho(\tilde{J}) = J$. Take a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, such that $\mathfrak{h}^+ = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{h}^- = \mathfrak{h} \cap \mathfrak{p}$. Next, choose a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}^{\mathbb{C}}$ such that $\mathfrak{k} = \mathfrak{g}_u \cap \mathfrak{g}$.

To complete the proof, it suffices to apply the argument in Theorem 5.2: representing M as $K/(K \cap Q)$, where K is the analytic subgroup of G corresponding to K, the element $a = \exp(\pi \tilde{J}) \in K \cap Q$ is a CR symmetry on M at the point $\underline{o}$, with respect to the restriction to M of any G_u-invariant Hermitian metric on $\mathscr{F}$. By the homogeneity of M with respect to K, as in Theorem 5.2 we conclude that M is a symmetric CR manifold. $\square$

Examples. 1. Consider the flag manifold $\mathfrak{F}$ consisting of the pairs (ℓ_1, ℓ_3) of linear subspaces ℓ_1, ℓ_3 of $\mathbb{C}^4$ with $\dim_{\mathbb{C}} \ell_1 = 1$, $\dim_{\mathbb{C}} \ell_3 = 3$ and $\ell_1 \subset \ell_3$. We identify $\mathbb{C}^4$ to the right vector space $\mathbb{H}^2$ over the division ring $\mathbb{H}$ of the quaternions. Consider the minimal orbit M of $\mathbf{SL}(2, \mathbb{H})$ in the flag manifold $\mathfrak{F}$.

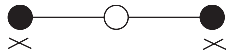
In this case we have

$$\mathfrak{g} = \left\{ \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ -\bar{z}_2 & \bar{z}_1 & -\bar{z}_4 & \bar{z}_3 \\ z_5 & z_6 & z_7 & z_8 \\ -\bar{z}_6 & \bar{z}_5 & -\bar{z}_8 & \bar{z}_7 \end{pmatrix} \middle| z_1, \dots, z_8 \in \mathbb{C},\ \mathscr{Re}(z_1 + z_7) = 0 \right\},$$

with

$$\mathfrak{q} = \left\{ \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ 0 & z_5 & z_6 & z_7 \\ 0 & z_8 & z_9 & z_{10} \\ 0 & 0 & 0 & z_{11} \end{pmatrix} \middle| z_1, \dots, z_{11} \in \mathbb{C},\ z_1 + z_5 + z_9 + z_{11} = 0 \right\}.$$

The corresponding σ-diagram is



One verifies that $(\mathfrak{g}, \mathfrak{q})$ has the (J) property with $\tilde{J}$ defined by the diagonal matrix:

$$\tilde{J} = \begin{pmatrix} -\frac{i}{2} & & & \\ & \frac{i}{2} & & \\ & & -\frac{i}{2} & \\ & & & \frac{i}{2} \end{pmatrix}.$$

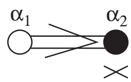
2. Consider the complex manifold $\mathscr{F}$ consisting of all isotropic 2-planes $V_2 \subset \mathbb{C}^2$, with respect to the nondegenerate symmetric bilinear form on $\mathbb{C}^5$ associated to the matrix $U := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. Due to the Witt's theorem, this is a $SO(5, \mathbb{C})$-homogeneous complex manifold, and the stabilizer of the standard 2-plane $\varrho := \langle e_1, e_2 \rangle$ is a parabolic subgroup. Hence $\mathscr{F}$ is a complex flag manifold. We consider the closed orbit M in $\mathscr{F}$ with respect to the action of the real form $SO(1,4)$ of $SO(5, \mathbb{C})$. We identify $SO(1,4)$ with the subgroup of $SO(5, \mathbb{C})$ consisting of those elements which preserve the Hermitian form represented by the matrix $K := \begin{pmatrix} & & 1 \\ & I_3 & \\ 1 & & \end{pmatrix}$. Hence M is the orbit of the point ϱ. The minimal parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ corresponding to M is given by

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & \zeta & b & \bar{\zeta} & 0 \\ \bar{z}_1 & i\lambda & -\bar{z}_2 & 0 & -\bar{\zeta} \\ t & z_2 & 0 & \bar{z}_2 & -b \\ \bar{z}_1 & 0 & -z_2 & -i\lambda & -\zeta \\ 0 & -z_1 & -t & -\bar{z}_1 & -a \end{pmatrix} \middle| \, z_1, z_2, \zeta \in \mathbb{C}, a, b, t, \lambda \in \mathbb{R} \right\}$$

and

$$\mathfrak{q} = \left\{ \left. \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & 0 \\ z_6 & z_7 & z_8 & 0 & -z_4 \\ 0 & 0 & 0 & -z_8 & -z_3 \\ 0 & 0 & 0 & -z_7 & -z_2 \\ 0 & 0 & 0 & -z_6 & -z_1 \end{pmatrix} \right| z_1, \dots, z_8 \in \mathbb{C} \right\}.$$

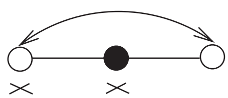
Its σ-diagram is



One checks that $(\mathfrak{g}, \mathfrak{q})$ has the (J) property with $\tilde{J}$ given by

$$\tilde{J} = \begin{pmatrix} 0 & & & & \\ & -i & & & \\ & & 0 & & \\ & & & i & \\ & & & & 0 \end{pmatrix}.$$

3. Consider the σ-diagram:



This diagram corresponds to the minimal orbit M of the standard action of $SU(1,3)$ on the complex flag manifold $\mathfrak{F}$ consisting of the pairs (ℓ_1, ℓ_2) of linear subspaces ℓ_1, ℓ_2 of $\mathbb{C}^4$ with $\dim_{\mathbb{C}} \ell_1 = 1$, $\dim_{\mathbb{C}} \ell_2 = 2$ and $\ell_1 \subset \ell_2$. We identify $SU(1,3)$ with the subgroup of $SL(4, \mathbb{C})$ consisting of the $\mathbb{C}$-linear transformations preserving the hermitian form represented by the matrix $K := \begin{pmatrix} & & 1 \\ & I_2 & \\ 1 & & \end{pmatrix}$. Hence we have $M = SU(1,3) \cdot \varrho$ where $\varrho := (\langle e_1 \rangle, \langle e_1, e_2 \rangle)$ is the standard flag. Accordingly, in this case

$$\mathfrak{g} = \{ A \in \mathfrak{sl}(4, \mathbb{C}) \mid A^* K + KA = 0 \} \cong \mathfrak{su}(1,3)$$

hence,

$$\mathfrak{g} = \left\{ \begin{pmatrix} w & z_4 & z_5 & is \\ -\bar{z}_3 & i\lambda & -\bar{z}_2 & -\bar{z}_4 \\ z_1 & z_2 & i\mu & -\bar{z}_5 \\ it & z_3 & -\bar{z}_1 & -\bar{w} \end{pmatrix} \middle| \begin{array}{c} z_1, z_2, z_3, z_4, z_5, w \in \mathbb{C} \\ s, t, \lambda, \mu \in \mathbb{R} \\ \lambda + \mu + 2\Im(w) = 0 \end{array} \right\}. \tag{4}$$

The parabolic subalgebra $\mathfrak{q} \subset \mathfrak{sl}(4, \mathbb{C})$ is

$$\mathfrak{q} = \left\{ \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ 0 & z_5 & z_6 & z_7 \\ 0 & 0 & z_8 & z_9 \\ 0 & 0 & z_{10} & z_{11} \end{pmatrix} \middle| z_1, \ldots, z_{11} \in \mathbb{C}, z_1 + z_5 + z_8 + z_{11} = 0 \right\}.$$

The minimal parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ has the (J) property with $\tilde{J}$ given by

$$\tilde{J} = \begin{pmatrix} -\frac{i}{4} & & & \\ & -\frac{i}{4} & & \\ & & \frac{3}{4}i & \\ & & & -\frac{i}{4} \end{pmatrix}.$$

We remark that in the three examples above $(\mathfrak{g}, \mathfrak{q})$ is not the minimal parabolic CR algebra associated to a Levi–Tanaka algebra. Indeed, the σ-diagram of a Levi–Tanaka algebra cannot contain crossed black nodes.

7. Necessary conditions for the (J) property

In this section, we show that a necessary condition for a transitive, minimal parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ to have the (J) property is that $\mathfrak{g}$ admits a $\mathbb{Z}_2$ gradation which is adapted to the CR structure (cf. Corollary 7.4). We also derive necessary conditions for the validity of the (J) property expressed in terms of the σ-diagram of $(\mathfrak{g}, \mathfrak{q})$. This information is the starting point for the complete classification of the fundamental minimal parabolic CR algebras having the (J) property, which is carried out in the next sections.

Let $(\mathfrak{g}, \mathfrak{q})$ be a minimal parabolic CR algebra. Let

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \cdots \oplus \mathfrak{g}^s$$

be its the decomposition into the direct sum of its simple ideals. Accordingly, we have a decomposition of $\mathfrak{q}$:

$$\mathfrak{q} = \mathfrak{q}^1 \oplus \mathfrak{q}^2 \oplus \cdots \oplus \mathfrak{q}^s,$$

in which each $\mathfrak{q}^i$ is a parabolic subalgebra of $(\mathfrak{g}^i)^{\mathbb{C}}$ and each $(\mathfrak{g}_i, \mathfrak{q}_i)$ is a minimal parabolic *CR* algebra; it will be called a *simple factor* of $(\mathfrak{g}, \mathfrak{q})$.

As a consequence of Theorem 3.1, $(\mathfrak{g}, \mathfrak{q})$ is fundamental if and only if each $(\mathfrak{g}_i, \mathfrak{q}_i)$ is fundamental. We also have:

Proposition 7.1. *A minimal parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ has the (J) property if and only if each of its simple factors $(\mathfrak{g}_i, \mathfrak{q}_i)$ has the (J) property.*

Proof. Indeed, if $\tilde{J}$ is the semisimple element with purely imaginary eigenvalues defining the complex structure on $\mathfrak{H}$, then we have a decomposition $\tilde{J} = \tilde{J}_1 + \cdots + \tilde{J}_s$ with $\tilde{J}_j \in \mathfrak{g}^{\mathrm{i}}_+$, and $[\tilde{J}, X] = [\tilde{J}_j, X]$ for all $X \in \mathfrak{g}^j$, for $j = 1, \dots, s$.

The statement is an easy consequence of this remark. $\square$

Hence the classification of the fundamental minimal parabolic *CR* algebras having the (J) property is reduced to the case where $\mathfrak{g}$ is simple.

Let us fix a maximally vectorial adapted Cartan subalgebra $\mathfrak{h}$ of $(\mathfrak{g}, \mathfrak{q})$, such that $\mathfrak{h}^+$ contains an element $\tilde{J}$ defining the complex structure of $\mathfrak{H}$. Let Δ be a σ-fundamental system of roots, such that, denoting by Σ^+ and Σ^- the positive and negative roots for Δ, we have $\Sigma^- \subset \mathcal{Q}$.

Every root α in Σ is a linear combination of the fundamental roots in Δ:

$$\alpha = \sum_{\beta \in \Delta} n_{\alpha,\beta} \beta, \tag{5}$$

where the coefficients $n_{\alpha,\beta} \in \mathbb{Z}$ are either all nonnegative or all nonpositive.

We set

$$\operatorname{supp}(\alpha) := \{\beta \in \Delta \mid n_{\alpha,\beta} \neq 0\}. \tag{6}$$

Let $\Delta_\times$ denote the set of fundamental roots that do not belong to $\mathcal{Q}$. Then

$$-\mathcal{Q}^n = \Sigma \backslash \mathcal{Q} = \{\beta \in \Sigma^+ \mid \operatorname{supp}(\beta) \cap \Delta_\times \neq \emptyset\}. \tag{7}$$

Consider the σ-diagram of $(\mathfrak{g}, \mathfrak{q})$ with respect to $\mathfrak{h}, \Delta$. We shall denote by $\Delta_\times$ the set of all fundamental roots that are crossed, by $\Delta_\bullet$, $\Delta_\circ$ the set of all black (resp. white) fundamental roots, and we also set

$$\operatorname{supp}_{\underset{\times}{\bullet}}(\alpha) := \operatorname{supp}(\alpha) \cap \Delta_\bullet \cap \Delta_\times; \ \ \operatorname{supp}_\circ(\alpha) := \operatorname{supp}(\alpha) \cap \Delta_\circ.$$

Recall that for each root $\alpha \in \Delta_\circ$ we have the conjugation formula

$$\bar{\alpha} = \varepsilon_C(\alpha) + \sum_{\beta \in \Delta_\bullet} n_{\bar{\alpha},\beta}\beta \text{ with } n_{\bar{\alpha},\beta} \in \mathbb{Z},\ n_{\bar{\alpha},\beta} \geqslant 0. \tag{8}$$

A subset $A \subset \Delta$ is said to be *connected* if the corresponding subset of the σ-diagram of $(\mathfrak{g}, \mathfrak{q})$ forms a connected subgraph of it.

We recall that a necessary and sufficient condition in order that $(\mathfrak{g}, \mathfrak{q})$ be fundamental is that (Theorem 3.1):

$$\alpha \in \Delta_\times \backslash \Sigma_\bullet \quad \Rightarrow \quad \varepsilon_C(\alpha) \notin \Delta_\times. \tag{9}$$

Proposition 7.2. *Let $(\mathfrak{g}, \mathfrak{q})$ be a simple fundamental transitive parabolic CR algebra with the (J) property, and let Δ be as above. Then, for the element $\tilde{J} \in \mathfrak{h}$ defining the complex structure on $\mathfrak{H}$, we have*:

$$\alpha(\tilde{J}) = \begin{cases} -i & \text{if } \alpha \in \mathcal{Q} \backslash \overline{\mathcal{Q}}, \\ 0 & \text{if } \alpha \in \mathcal{Q}^r \cap \overline{\mathcal{Q}}^r, \\ i & \text{if } \alpha \in \overline{\mathcal{Q}} \backslash \mathcal{Q}. \end{cases} \tag{10}$$

Proof. The fact that $\tilde{J}$ defines the complex structure of $\mathfrak{H} = \mathfrak{H}_+/\mathfrak{g}_+$ means that $X + i[\tilde{J}, X] \in \mathfrak{q}$ and $X - i[\tilde{J}, X] \in \overline{\mathfrak{q}}$ for all $X \in \mathfrak{H}_+$.

This implies that $\alpha(\tilde{J})$ is equal to i on $\overline{\mathcal{Q}} \backslash \mathcal{Q}$ and to $-i$ on $\mathcal{Q} \backslash \overline{\mathcal{Q}}$, respectively.

Moreover, since $\tilde{J}$ belongs to the radical of $\mathfrak{g}_+$, we have $[\tilde{J}, Z] = 0$ for each $X \in \mathfrak{q}^r \cap \overline{\mathfrak{q}}^r$, and hence $\alpha(\tilde{J}) = 0$ if $\mathfrak{g}^\alpha \subset \mathfrak{q}^r \cap \overline{\mathfrak{q}}^r$, i.e. $\alpha \in \mathcal{Q}^r \cap \overline{\mathcal{Q}}^r$. □

Proposition 7.3. *Assume that the simple fundamental transitive parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ has the (J) property. Let Δ be a σ-fundamental system of roots. Then*

$$\alpha(\tilde{J}) = \begin{cases} i & \text{if } \begin{cases} \alpha \in \Delta_\circ \cap \Delta_\times & \text{and} \quad \operatorname{supp}(\bar{\alpha}) \cap \Delta_\times = \emptyset \quad \text{or} \\ \alpha \in \Delta_\times \cap \Delta_\bullet; \end{cases} \\ 0 & \text{if } \begin{cases} \alpha \in \Delta_\circ \backslash \Delta_\times & \text{and} \quad \operatorname{supp}(\bar{\alpha}) \cap \Delta_\times = \emptyset, \quad \text{or} \\ \alpha \in \Delta_\circ \cap \Delta_\times & \text{and} \quad \operatorname{supp}(\bar{\alpha}) \cap \Delta_\times \neq \emptyset, \quad \text{or} \\ \alpha \in \Delta_\bullet \backslash \Delta_\times; \end{cases} \\ -i & \text{if } \alpha \in \Delta_\circ \backslash \Delta_\times \quad \text{and} \quad \operatorname{supp}(\bar{\alpha}) \cap \Delta_\times \neq \emptyset. \end{cases} \tag{11}$$

Proof. We utilize Proposition 7.2.

First we consider an $\alpha \in \Delta_\bullet$:

if $\alpha \in \Delta_\times$, then $\alpha \in \overline{\mathcal{Q}} \backslash \mathcal{Q}$ and $\alpha(\tilde{J}) = i$;

if $\alpha \notin \Delta_\times$, then $\alpha \in \mathcal{Q}^r \cap \overline{\mathcal{Q}}^r$ and $\alpha(\tilde{J}) = 0$.

Next we consider an $\alpha \in \Delta_\circ$.

Assume first that $\mathrm{supp}(\bar{\alpha}) \cap \Delta_\times = \emptyset$: this means that $\alpha \in \bar{\mathcal{Q}}$:

if $\alpha \notin \Delta_\times$, then $\pm\alpha$, $\pm\bar{\alpha} \in \mathcal{Q}$, hence $\alpha \in \mathcal{Q}^r \cap \bar{\mathcal{Q}}^r$ and $\alpha(\tilde{J}) = 0$;

if $\alpha \in \Delta_\times$, then $\alpha \in \bar{\mathcal{Q}} \backslash \mathcal{Q}$ and $\alpha(\tilde{J}) = i$.

Finally, assume that $\alpha \in \Delta_\circ$ and $\mathrm{supp}(\bar{\alpha}) \cap \Delta_\times \neq \emptyset$: this means that $\alpha \notin \bar{\mathcal{Q}}$:

if $\alpha \notin \Delta_\times$, we have $\alpha \in \mathcal{Q} \backslash \bar{\mathcal{Q}}$, and hence $\alpha(\tilde{J}) = -i$;

if $\alpha \in \Delta_\times$, since $\mathrm{supp}(\bar{\alpha}) \cap \Delta_\times \neq \emptyset$, neither α nor $\bar{\alpha}$ belong to $\mathcal{Q} \cup \overline{\mathcal{Q}}$. In this case $\varepsilon_C(\alpha) \in \mathcal{Q} \backslash \overline{\mathcal{Q}}$, because $(\mathfrak{g}, \mathfrak{q})$ is fundamental, and hence $\varepsilon_C(\alpha)(\tilde{J}) = -i$. The segment E joining α with $\varepsilon_C(\alpha)$ is a sub-diagram of type (A_ℓ) and $E \backslash \{\alpha, \varepsilon_C(\alpha)\} \subset \Delta_\bullet$. There is exactly one root β in $E \cap \Delta_\times \cap \Delta_\bullet$, because otherwise $\overline{\mathcal{Q}} \backslash \mathcal{Q}$ would contain a root γ with $\gamma(\tilde{J}) = 2i$. Since $\mu = \sum_{\tau \in E \backslash \{\varepsilon_C(\alpha)\}} \tau$ and $\alpha + \mu$ both belong to $\overline{\mathcal{Q}} \backslash \mathcal{Q}$, from $\mu(\tilde{J}) = (\alpha + \mu)(\tilde{J}) = i$, we obtain that $\alpha(\tilde{J}) = 0$. $\square$

Corollary 7.4. *Assume that the simple fundamental transitive parabolic CR algebra* $(\mathfrak{g}, \mathfrak{q})$ *has the* (J) *property. Then* $(\mathfrak{g}, \mathfrak{q})$ *admits a compatible inner* $\mathbb{Z}_2$ *gradation*

$$\mathfrak{g} = \mathfrak{g}_{[0]} \oplus \mathfrak{g}_{[1]}, \quad \mathfrak{q} = \mathfrak{q}_{[0]} \oplus \mathfrak{q}_{[1]}.$$

This means that there exists an inner involutive automorphism $\vartheta : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ *such that*:

$$\begin{cases} \vartheta(\mathfrak{g}) = \mathfrak{g} \\ \vartheta(\mathfrak{q}) = \mathfrak{q} \\ \mathfrak{q}_{[0]} = \{Z \in \mathfrak{q} \mid \vartheta(Z) = Z\} \subset \mathfrak{q} \cap \bar{\mathfrak{q}}. \end{cases} \tag{12}$$

Proof. In fact we can take the $\mathbb{Z}_2$-gradation defined by the inner automorphism $\vartheta = Ad(\exp(\pi \tilde{J}))$. $\square$

Corollary 7.5. *Let* $(\mathfrak{g}, \mathfrak{q})$ *be a fundamental minimal parabolic CR algebra. Fix an adapted maximally vectorial Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$, *and a* σ*-fundamental system of roots* Δ *such that* $\Sigma^- \subset \mathcal{Q}$. *Consider the* $\mathbb{R}$*-linear map* $\lambda : \mathfrak{h}^*_{\mathbb{R}} \to \mathbb{C}$ *determined by*

$$\lambda(\alpha) = \begin{cases} i & \textit{if} \begin{cases} \alpha \in \Delta_\circ \cap \Delta_\times & \textit{and} \quad \mathrm{supp}(\bar{\alpha}) \cap \Delta_\times = \emptyset \quad \textit{or} \\ \alpha \in \Delta_\times \cap \Delta_\bullet; \end{cases} \\ 0 & \textit{if} \begin{cases} \alpha \in \Delta_\circ \backslash \Delta_\times & \textit{and} \quad \mathrm{supp}(\bar{\alpha}) \cap \Delta_\times = \emptyset, \quad \textit{or} \\ \alpha \in \Delta_\circ \cap \Delta_\times & \textit{and} \quad \mathrm{supp}(\bar{\alpha}) \cap \Delta_\times \neq \emptyset, \quad \textit{or} \\ \alpha \in \Delta_\bullet \backslash \Delta_\times; \end{cases} \\ -i & \textit{if} \quad \alpha \in \Delta_\circ \backslash \Delta_\times \quad \textit{and} \quad \mathrm{supp}(\bar{\alpha}) \cap \Delta_\times \neq \emptyset. \end{cases} \tag{13}$$

Then the following properties are equivalent:

(a) $(\mathfrak{g}, \mathfrak{q})$ *has the* (J) *property*;

(b) *For each* $\alpha \in \Sigma$, $\lambda(\bar{\alpha}) = \overline{\lambda(\alpha)}$, *and* $\lambda(\alpha) = i$ $\forall \alpha \in \overline{\mathcal{Q}} \backslash \mathcal{Q}$.

Proof. $(b) \Rightarrow (a)$: Since Δ is a basis of $\mathfrak{h}^*_{\mathbb{R}}$, there is a unique vector $Z \in \mathfrak{h}^{\mathbb{C}}$ such that

$$\lambda(\alpha) = \alpha(Z) \quad \forall \alpha \in \Sigma.$$

Since λ commutes with the conjugation, $Z \in \mathfrak{h}$. Moreover, the condition $\lambda = i$ on $\bar{\mathcal{Q}} \backslash \mathcal{Q}$ implies that $X + i[Z, X] \in \mathfrak{q}$ for each $X \in \mathfrak{H}_+$, which in turns implies that $\rho(Z) = J$, proving (a).

$(a) \Rightarrow (b)$: According to Corollary 6.3, we can assume that $\mathfrak{h}$ contains an element $\tilde{J} \in \mathfrak{h}$ satisfying $\rho(\tilde{J}) = J$. The statement follows directly from (11) and (10). $\square$

8. Classification of fundamental minimal parabolic CR algebras having the (J) property

We give now our main result, leading to a complete classification of the fundamental minimal parabolic CR algebras having the (J) property. Our criteria involve the σ-diagram of $(\mathfrak{g}, \mathfrak{q})$.

Theorem 8.1. *Let* $(\mathfrak{g}, \mathfrak{q})$ *be a transitive simple fundamental minimal parabolic CR algebra, which is neither totally real nor totally complex. Fix an adapted maximally vectorial Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$, *and a* σ*-fundamental system of roots* Δ *such that* $\Sigma^- \subset \mathcal{Q}$.

A necessary and sufficient condition in order that $(\mathfrak{g}, \mathfrak{q})$ *has the* (J) *property is that the following conditions* (A)–(C) *are satisfied*:

(A) *If* $\alpha, \varepsilon_C(\alpha) \in \Delta_\circ \backslash \Delta_\times$ *and* $\operatorname{supp}(\bar{\alpha}) \cap \Delta_\times \neq \emptyset$, *then*

$$\sum_{\beta \in \operatorname{supp}_{\underset{\times}{\bullet}}(\bar{\alpha})} n_{\bar{\alpha},\beta} = 2.$$

(B) *If* $A \subset \Delta \backslash \Delta_\times$ *is connected, then* $\#(\varepsilon_C(A \cap \Delta_\circ) \cap \Delta_\times) \leqslant 1$.

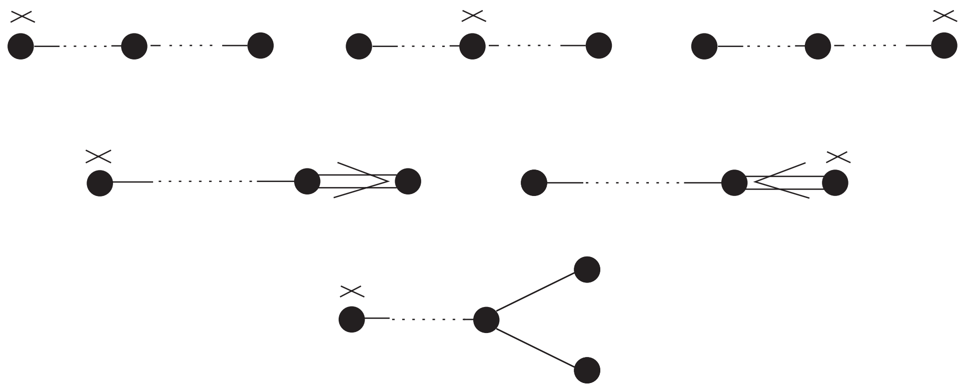
(C) *If* $B \subset \Delta_\bullet$ *is connected and* $\alpha = \sum_{\beta \in B} n_{\alpha,\beta} \beta \in \Sigma$, *then*

$$\sum_{\beta \in B \cap \Delta_\times} |n_{\alpha,\beta}| \leqslant 1.$$

When $\Delta_{\underset{\times}{\bullet}} = \emptyset$, *conditions* (A) *and* (C) *are vacuously satisfied*: *property* (J) *is equivalent to condition* (B), *which in turns means that* $(\mathfrak{g}, \mathfrak{q})$ *is the CR algebra corresponding to a semisimple Levi–Tanaka algebra.*

The minimal parabolic CR algebras whose σ*-diagram satisfies* (A)–(C) *and* $\Delta_{\underset{\times}{\bullet}} \neq \emptyset$ *are listed in Table* 3.

Remarks. (I) An equivalent formulation of condition (B) is: for each $\alpha \in \Delta_\bullet \cap \Delta_\times$, the connected component of α in $\Delta_\bullet$ is one of the following:



(II) The minimal parabolic *CR* algebras with the (J) property which are not Levi–Tanaka are those corresponding to σ-diagrams satisfying (A)–(C) above and containing crossed black nodes. Hence Table 3 provides a classification of the minimal orbits in complex flag manifolds which are symmetric *CR* manifolds in the sense specified by Theorem 6.4, and which are not standard *CR* manifolds in the sense of [11].

Proof of Theorem 8.1. *Necessity*: Assume that $(\mathfrak{g}, \mathfrak{q})$ has the (J) property. We shall verify the validity of conditions (A)–(C) by making use of the function $\lambda : \Sigma \to \mathbb{C}$ introduced in Corollary 7.5. We know that λ commutes with the conjugation $\alpha \to \bar{\alpha}$ and satisfies $\lambda(\alpha) = i$ for each $\alpha \in \bar{\mathcal{Q}} \backslash \mathcal{Q}$.

We discuss (A). Assume that $\alpha, \varepsilon_C(\alpha) \in \Delta_\circ \backslash \Delta_\times$ and $\mathrm{supp}(\bar{\alpha}) \neq \emptyset$. From (8) we have

$$\lambda(\bar{\alpha}) = \lambda(\varepsilon_C(\alpha)) + i \cdot \sum_{\beta \in \mathrm{supp}_{\substack{\bullet \\ \times}}(\bar{\alpha})} n_{\bar{\alpha},\beta}, \tag{14}$$

which can be rewritten as

$$\overline{\lambda(\alpha)} = \lambda(\varepsilon_C(\alpha)) + i \cdot \sum_{\beta \in \mathrm{supp}_{\substack{\bullet \\ \times}}(\bar{\alpha})} n_{\bar{\alpha},\beta}. \tag{15}$$

The conjugation formula (8) also yields:

$$\overline{\varepsilon_C(\alpha)} = \alpha + \sum_{\beta \in \Delta_\bullet} n_{\bar{\alpha},\beta} \beta. \tag{16}$$

Since $\mathrm{supp}_{\substack{\bullet \\ \times}}(\bar{\alpha}) \neq \emptyset$, we have $\lambda(\alpha) = \lambda(\varepsilon_C(\alpha)) = -i$. Then from (15) we get $\sum_{\beta \in \Delta_{\substack{\bullet \\ \times}}} n_{\bar{\alpha},\beta} = 2$, yielding (A).

To prove (B), we assume by contradiction that there exists a segment $A \subset \Delta \backslash \Delta_\times$ with end nodes $\alpha_1, \alpha_2 \in \Delta_\circ$, and $\varepsilon_C(\alpha_1), \varepsilon_C(\alpha_2) \in \Delta_\times$. Let $\alpha = \sum_{\beta \in A} \beta$. We have $\alpha \in \mathcal{Q} \backslash \bar{\mathcal{Q}}$, and hence $\lambda(\alpha) = -i$. On the other hand, $i\lambda(\beta) \geqslant 0$ for all $\beta \in A$, and hence $i\lambda(\alpha_1) + i\lambda(\alpha_2) \leqslant i\lambda\left(\sum_{\beta \in A} \beta\right) = 1$. But this gives a contradiction because $i\lambda(\alpha_1) = i\lambda(\alpha_2) = 1$.

Finally we prove (C). Assume $B \subset \Delta_\bullet$ connected, $B \cap \Delta_\times \neq \emptyset$, and that $\alpha = \sum_{\beta \in B} n_{\alpha,\beta} \beta \in \Sigma^+$. Then $\alpha \in \bar{\mathcal{Q}} \backslash \mathcal{Q}$. Whence

$$i = \lambda(\alpha) = \left(\sum_{\beta \in B \cap \Delta_\times} n_{\alpha,\beta} \right) \cdot i$$

and this yields the conclusion.

Sufficiency. We distinguish two cases:

Case I. $\Delta_\bullet \cap \Delta_\times = \emptyset$.

Case II. $\Delta_\bullet \cap \Delta_\times \neq \emptyset$.

Case I: We prove that $\mathfrak{g}$ is a semisimple Levi–Tanaka algebra, and that $(\mathfrak{g}, \mathfrak{q})$ is the corresponding minimal parabolic *CR* algebra according to Section 5, namely

$$\mathfrak{q} = \mathfrak{m}^{01} \oplus \mathfrak{g}_+^{\mathbb{C}}.$$

It is known (cf. [5]) that the $\mathbb{Z}$-gradations of $\mathfrak{g}$ are classified by means of weighted Satake diagrams. These are obtained from the Satake diagram of $\mathfrak{g}$ by attaching a nonpositive integer to each node corresponding to a fundamental root, with the rule that black roots have weight zero, and white roots joined by an arrow have the same weight. Thus the $\mathbb{Z}$-gradations of $\mathfrak{g}$ correspond to pairs (Δ, g) where Δ is a σ-fundamental system of roots in Σ and $g : \Delta \to \mathbb{Z}$ satisfies

$$g(\alpha) \leqslant 0 \ \forall \alpha \in \Delta, \quad g(\Delta_\bullet) = \{0\}, \ g(\alpha) = g(\varepsilon_C(\alpha)) \quad \forall \alpha \in \Delta_\circ.$$

The weight map g uniquely extends to Σ by $\mathbb{Z}$-linearity.

Now assume that the σ-diagram of $(\mathfrak{g}, \mathfrak{q})$ satisfies (B) and $\Delta_{\underset{\times}{\bullet}} = \emptyset$. Define a map $g : \Delta \to \{-1, 0\}$ as follows:

$$g(\alpha) = \begin{cases} -1 & \text{if } \alpha \in \Delta_\times \text{ or } \varepsilon_C(\alpha) \in \Delta_\times, \\ 0 & \text{if } \{\alpha, \varepsilon_C(\alpha)\} \cap \Delta_\times = \emptyset. \end{cases} \tag{17}$$

From the assumption $\Delta_{\underset{\times}{\bullet}} = \emptyset$ it follows that (Δ, g) determines a $\mathbb{Z}$-gradation

$$\mathfrak{g} = \bigoplus_{k=-\mu}^{\mu} \mathfrak{g}_k$$

according to the above remarks. This gradation is subordinated to the $\mathbb{Z}$-gradation of $\mathfrak{g}^{\mathbb{C}}$ defined by

$$\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Sigma_0} \mathfrak{g}_k, \quad \mathfrak{g}_k^{\mathbb{C}} = \bigoplus_{\alpha \in \Sigma_k} \mathfrak{g}^{\alpha} \quad \text{for } k \neq 0,$$

where for each $k \in \mathbb{Z}$, we set $\Sigma_k \coloneqq \{\alpha \in \Sigma \mid g(\alpha) = k\}$.

Lemma 8.2. *Setting* $\Sigma_{\geqslant 0} \coloneqq \bigcup_{p \geqslant 0} \Sigma_p$, *we have*

$$\Sigma_{-1} = (\mathscr{Q} \backslash \overline{\mathscr{Q}}) \cup (\overline{\mathscr{Q}} \backslash \mathscr{Q}), \tag{18}$$

$$\mathscr{Q} \cap \overline{\mathscr{Q}} \subset \Sigma_{\geqslant 0} \subset \mathscr{Q}. \tag{19}$$

Proof. To prove (18), assume $\alpha \in \Sigma_{-1}$. Then $\alpha \in \Sigma^+$ and there exists a unique $\beta \in \operatorname{supp}(\alpha) \cap \Delta_\circ$ such that $g(\alpha) = -1$ and $\operatorname{supp}(\alpha) \backslash \{\beta\} \subset \Sigma_0$. If $\beta \in \Delta_\times$, we have $\alpha \notin \mathscr{Q}$ and $\operatorname{supp}(\bar{\alpha}) \cap \Delta_\times = \emptyset$ whence $\alpha \in \overline{\mathscr{Q}} \backslash \mathscr{Q}$. If $\varepsilon_C(\beta) \in \Delta_\times$, we get $\operatorname{supp}(\alpha) \cap \Delta_\times = \emptyset$ and since $\varepsilon_C(\beta) \in \operatorname{supp}(\bar{\alpha}) \cap \Delta_\times$, we have $\alpha \in \mathscr{Q} \backslash \overline{\mathscr{Q}}$. We have proved the inclusion $\subset$ in (18). Now let $\alpha \in \overline{\mathscr{Q}} \backslash \mathscr{Q}$. Then, according to property (B) applied to $\operatorname{supp}(\alpha)$, there exists a unique $\beta \in \operatorname{supp}(\alpha) \cap \Delta_\circ$ such that $\varepsilon_C(\beta) \in \Delta_\times$. Hence by definition $g(\alpha) = -1$. We have proved that $\overline{\mathscr{Q}} \backslash \mathscr{Q} \subset \Sigma_{-1}$. Since g commutes with the conjugation, we obtain also that $\mathscr{Q} \backslash \overline{\mathscr{Q}} \subset \Sigma_{-1}$. This concludes the proof of (18).

Let $\alpha \in \mathscr{Q} \cap \overline{\mathscr{Q}}$. To prove that $g(\alpha) \geqslant 0$ it suffices to consider the case where $\alpha \in \Sigma^+$. Accordingly, we have $\operatorname{supp}(\alpha) \cap \Delta_\times = \operatorname{supp}(\bar{\alpha}) \cap \Delta_\times = \emptyset$, and by definition we get $g(\alpha) = 0$. Hence $\mathscr{Q} \cap \overline{\mathscr{Q}} \subset \Sigma_{\geqslant 0}$. To conclude the proof, observe that $\Sigma_{\geqslant 0} \subset \Sigma^- \subset \mathscr{Q}$. $\square$

We remark that (18) implies that $\mu \geqslant 2$ because otherwise we would have $\mathfrak{g}^{\mathbb{C}} = \mathfrak{q} + \bar{\mathfrak{q}}$, i.e. $(\mathfrak{g}, \mathfrak{q})$ would be a totally complex *CR* algebra.

Now we consider the $\mathbb{C}$-linear subspace $\mathfrak{m}^{01}$ of $\mathfrak{g}_{-1}^{\mathbb{C}}$ defined by

$$\mathfrak{m}^{01} \coloneqq \bigoplus_{\alpha \in \mathscr{Q} \backslash \overline{\mathscr{Q}}} \mathfrak{g}^{\alpha}.$$

It follows from (19) that

$$\mathfrak{q} = \mathfrak{m}^{01} \oplus \mathfrak{g}_+^{\mathbb{C}}.$$

Moreover, according to (18) we have

$$\mathfrak{g}_{-1}^{\mathbb{C}} = \mathfrak{m}^{01} \oplus \overline{\mathfrak{m}^{01}}.$$

Hence there exists a unique complex structure $J : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ such that

$$\mathfrak{m}^{01} = \{X + iJX | X \in \mathfrak{g}_{-1}\}.$$

As a consequence of (18) and (19) we have

$$[\mathfrak{m}^{01}, \mathfrak{m}^{01}] = 0 \quad \text{and} \quad [\mathfrak{g}_0, \mathfrak{m}^{01}] \subset \mathfrak{m}^{01}.$$

This implies that J is a partial complex structure on the $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ according to the definition in Section 5.

Since $\mathfrak{g}$ is simple, according to [8, Theorem 3.21], to prove that $(\mathfrak{g}, J)$ is a Levi–Tanaka algebra, it suffices to show that it is fundamental and transitive. The first property follows from the fact that $\Delta \subset \Sigma_0 \cup \Sigma_{-1}$. The second one is true since $\mu \geqslant 2$ (cf. Lemma 3.17 in [8]). This completes the proof in Case I).

Case II: We shall determine all the σ-diagrams which contain black crossed knotes and satisfy properties (A)–(C).

Proposition 8.3. *Let* $(\mathfrak{g}, \mathfrak{q})$ *be a simple, transitive fundamental minimal parabolic CR algebra. Assume that the* σ*-diagram of* $(\mathfrak{g}, \mathfrak{q})$ *satisfies conditions* (A)–(C).

Table 2 *contains the list of all possible Araki's elementary conjugation sub-diagrams* S (*cf. Table* 1) *of the* σ*-diagram of* $(\mathfrak{g}, \mathfrak{q})$ *that contain some crossed knot.*

Proof. We consider the various possibilities for S according to Table 1.

(A_1): since S contains crossed knots this possibility is excluded by (9).

$(A_1) \times (A_1)$: by (9), S contains exactly one crossed root and is diagram 1 in Table 2.

(A_3): by (9) α_2 is not crossed; hence by property (A) both α_1 and α_3 must be crossed. Hence S is diagram 2 in Table 2.

(A_l): by (C), $S \cap \Delta_\bullet$ contains at most one crossed root. If $S \cap \Delta_{\underset{\times}{\bullet}} = \emptyset$, we have diagrams 3 in Table 2. Otherwise, by (A) either α_1 or α_l is crossed and by (9) we obtain diagram 4 in Table 2.

(B_l): set $B := \{\alpha_k \mid k \geqslant 2\} = S \cap \Delta_\bullet$. Since α_1 cannot be crossed, according to (C) applied to B, we have that S contains exactly one crossed root $\beta \in \Delta_\bullet$. Actually the only possibility is $\beta = \alpha_2$. Indeed, $\gamma := \alpha_2 + 2\sum_{j=3}^{l} \alpha_j$ is a root. Since $\mathrm{supp}(\gamma) \subset B$ and $n_{\gamma,\alpha_j} = 2$ for $j \geqslant 3$, no α_j with $j \geqslant 3$ can be crossed, because of (C). Thus we obtain diagram 5 in Table 2.

(C_l): set $B = \{\alpha_s \mid s \geqslant 3\} \subset S \cap \Delta_\bullet$. Then by (C), B contains at most one crossed root. Moreover, by applying condition (A) to the white root α_2, we see that B must contain a crossed root β, otherwise we would have $S \cap \Delta_\times = \{\alpha_1\}$ and

$$\sum_{\gamma \in \mathrm{supp}_{\underset{\times}{\bullet}}(\bar{\alpha}_2)} n_{\bar{\alpha}_2,\gamma} = n_{\bar{\alpha}_2,\alpha_1} = 1.$$

Since $2(\alpha_3 + \cdots + \alpha_{l-1}) + \alpha_l$ is a root, (C) yields $\beta = \alpha_l$. Using again (A) for α_2, we conclude that also α_1 is crossed and obtain diagram 6 in Table 2.

Table 1
Elementary conjugation diagrams [2]

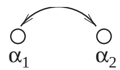
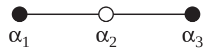
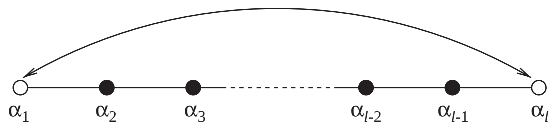
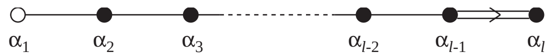
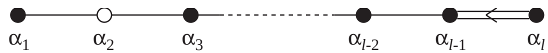
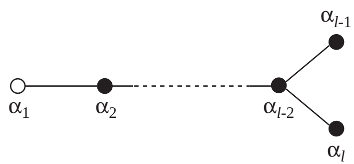
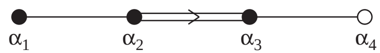
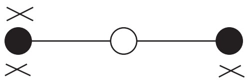
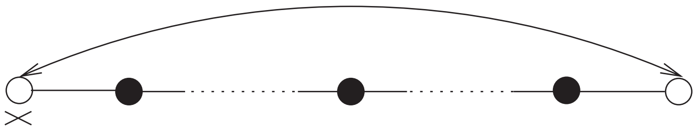
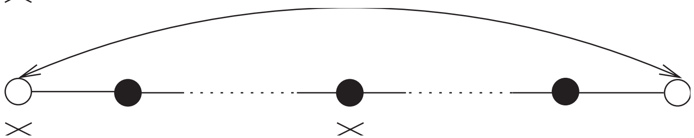
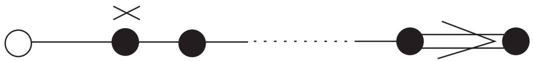
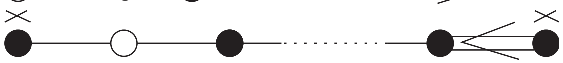
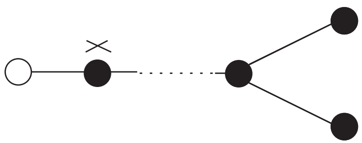
A_1		$\bar{\alpha}_1 = \alpha_1$
$A_1 \times A_1$		$\bar{\alpha}_1 = \alpha_2$
A_3		$\bar{\alpha}_2 = \alpha_1 + \alpha_2 + \alpha_3$
A_ℓ		$\bar{\alpha}_1 = \alpha_2 + \cdots + \alpha_\ell$ $\bar{\alpha}_\ell = \alpha_1 + \cdots + \alpha_{\ell-1}$
B_ℓ		$\bar{\alpha}_1 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_\ell)$
C_ℓ		$\bar{\alpha}_2 = \alpha_1 + \alpha_2 + 2(\alpha_3 + \cdots + \alpha_{\ell-1}) + \alpha_\ell$
D_ℓ		$\bar{\alpha}_1 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{\ell-2}) + \alpha_{\ell-1} + \alpha_\ell$
F_4		$\bar{\alpha}_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$

Table 2
Elementary conjugation sub-diagrams in σ-diagrams of minimal parabolic CR algebras with the (J) property

1		And the analogous with a cross on the other white root	$A_1 \times A_1$
2			A_3
3		And the analogous with a cross on the other white root	A_l
4		And the analogous with the crossed white root on the right	A_l
5			B_l
6			C_l
7			D_l

(D_l): S contains exactly one crossed black root β. This cannot be α_{l-1} or α_l because of condition (A) for α_1. Since $\gamma = \alpha_2 + 2\sum_{j=3}^{l-2} + \alpha_{l-1} + \alpha_l$ is a root, by (C) we have $\beta = \alpha_2$, and we obtain diagram 7 in Table 2.

(F_4): there are no subdiagrams S corresponding to F_4. Indeed, S should contain just one black crossed root β. By condition (A) for α_4, we should have $\beta = \alpha_2$. But $\gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3$ is a root with $\mathrm{supp}(\gamma) \subset \Delta_\bullet$ and $\mathrm{supp}(\gamma) \cap \Delta_\times = \{\alpha_2\}$, and this would contradict (C). $\square$

Corollary 8.4. *Table 3 lists all the σ-diagrams with $\Delta_\bullet \cap \Delta_\times \neq \emptyset$ which satisfy conditions* (A)–(C).

Proof. This follows by imposing that the Araki's elementary conjugations subdiagrams of the σ-diagram are those from Table 2 and taking into account the properties (A)–(C). $\square$

Now we are in a position to complete the proof of Theorem 8.1. It remains to prove that the minimal parabolic CR algebras whose σ-diagram is one of those in Table 3 all have the (J) property.

We accomplish this by verifying that the function $\lambda : \Delta \to \mathbb{C}$ defined in Corollary 7.5 commutes with the conjugation $\alpha \to \bar{\alpha}$ and satisfies $\lambda = i$ on $\overline{\mathcal{Q}} \backslash \mathcal{Q}$.

To verify the first fact, we remark that $\overline{\lambda(\alpha)} = \lambda(\bar{\alpha})$ holds true if $\bar{\alpha} = -\alpha$, because λ is linear and takes purely imaginary values. Hence we consider the case where $\alpha \in \Delta_\circ$. Set $S = \{\alpha\} \cup \mathrm{supp}(\bar{\alpha})$. If $S \cap \Delta_\times = \emptyset$, then $\lambda(\alpha) = \lambda(\bar{\alpha}) = 0$. Otherwise S consists of the knotes belonging to one of the elementary conjugation diagrams in Table 2, and the fact that $\overline{\lambda(\alpha)} = \lambda(\bar{\alpha})$ can be directly read off these diagrams.

To prove that $\lambda = i$ on $\overline{\mathcal{Q}} \backslash \mathcal{Q}$, we make the following

Claim. *Each σ-diagram in Table* 3 *has the following property*:

(D) *Let $A \subset \Delta$ be a connected set such that $A \cap \Delta_\times = \emptyset$ and $A \cap \lambda^{-1}(-i) \neq \emptyset$. Then $\# A \cap \lambda^{-1}(-i) = 1$ and the unique root $\alpha \in \Sigma^+$ with* $\mathrm{supp}(\alpha) = A$ *is $\alpha = \sum_{\beta \in A} \beta$.*

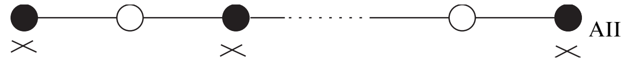
(C) and (D) actually imply that $\lambda = i$ on $\overline{\mathcal{Q}} \backslash \mathcal{Q}$. Indeed, fix $\alpha \in \overline{\mathcal{Q}} \backslash \mathcal{Q}$. Then $\alpha \in \Sigma^+$ and $\mathrm{supp}(\alpha) \cap \Delta_\times \neq \emptyset$. Since $\bar{\alpha} \in \mathcal{Q}$, we have either $\bar{\alpha} \in \Sigma^-$ or $\bar{\alpha} \in \Sigma^+$ with $\mathrm{supp}(\bar{\alpha}) \cap \Delta_\times = \emptyset$. In the first case, $\alpha \in \Sigma_\bullet$; setting $B = \mathrm{supp}(\alpha)$ we have

$$\lambda(\alpha) = \sum_{\gamma \in B \cap \Delta_\times} n_{\alpha,\gamma} \lambda(\gamma) = i$$

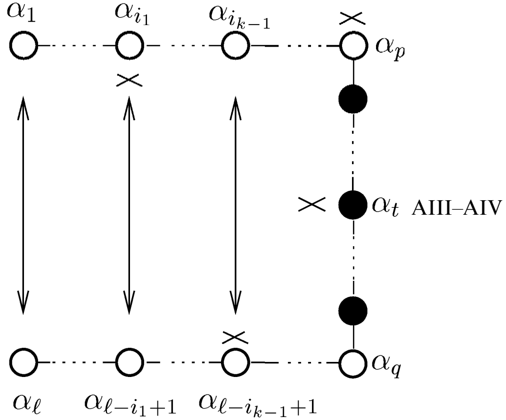
by (C). If $\bar{\alpha} \in \Sigma^+$, then $\mathrm{supp}(\bar{\alpha}) \cap \Delta_\bullet = \emptyset$, and $\mathrm{supp}(\bar{\alpha})$ contains at least one root β such that $\lambda(\beta) = -i$ because $\mathrm{supp}(\alpha) \cap \Delta_\times \neq \emptyset$. Hence, by applying (D) to $A = \mathrm{supp}(\bar{\alpha})$ we get $A \cap \lambda^{-1}(-i) = \{\beta\}$ and $\bar{\alpha} = \sum_{\gamma \in A} \gamma$, whence $\lambda(\bar{\alpha}) = -i$. This implies that $\lambda(\alpha) = i$ because λ commutes with the conjugation.

Table 3
σ-diagrams of minimal parabolic *CR* algebras having the (J) property, with $\Delta_\bullet \cap \Delta_\times \neq \emptyset$

1

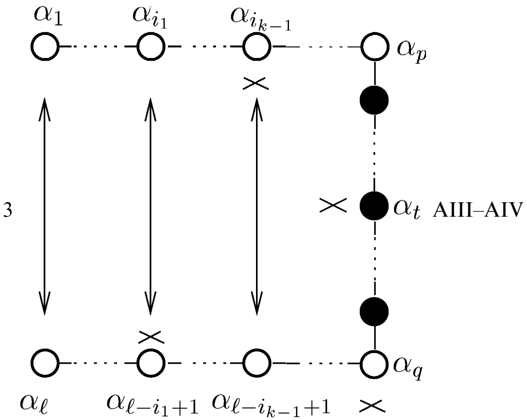


2



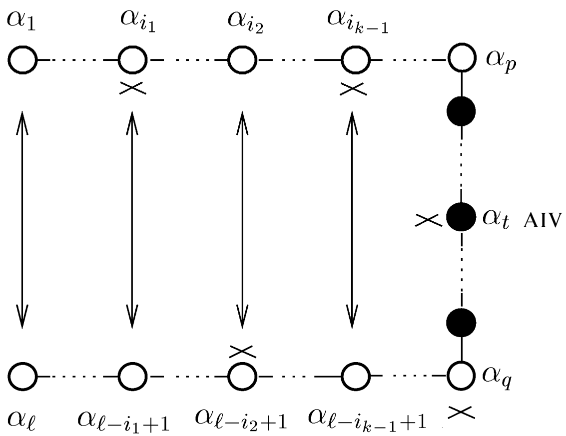
$\Delta_\times \cap \Delta_\circ = \alpha_{i_1}, \ldots, \alpha_{\ell-i_{k-1}+1}, \alpha_{i_k}$
$1 \leqslant i_1 < \cdots < i_k = p, \quad k$ odd, $k \geqslant 1$

3



$\Delta_\times \cap \Delta_\circ = \alpha_{\ell-i_1+1}, \ldots, \alpha_{i_{k-1}}, \alpha_{\ell-i_k+1}$
$1 \leqslant i_1 < \cdots < i_k = p, \quad k$ odd, $k \geqslant 1$

4



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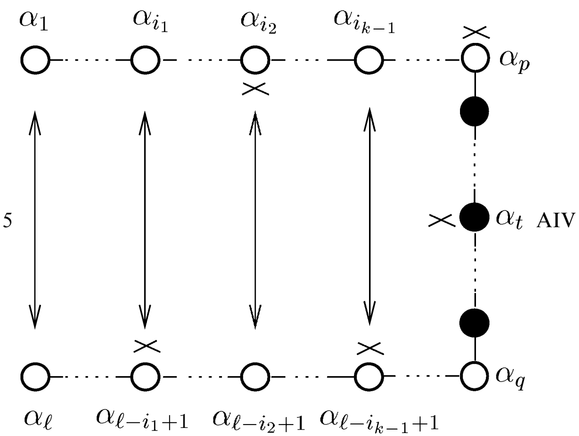
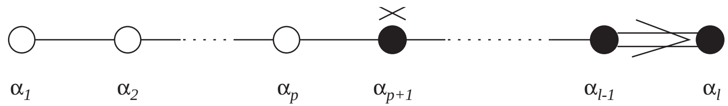
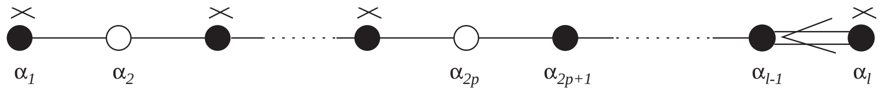


$\varDelta_\times \cap \varDelta_\circ = \alpha_{i_1}, \alpha_{\ell-i_2+1}, \ldots, \alpha_{i_{k-1}}, \alpha_{\ell-i_k+1}$
$1 \leqslant i_1 < i_2 < \cdots < i_k = p, \quad k \text{ even}, \quad k \geqslant 2$

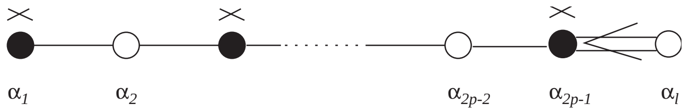
$\varDelta_\times \cap \varDelta_\circ = \alpha_{\ell-i_1+1}, \alpha_{i_2}, \ldots, \alpha_{\ell-i_{k-1}+1}, \alpha_{i_k}$
$1 \leqslant i_1 < i_2 < \cdots < i_k = p, \quad k \text{ even}, \quad k \geqslant 2$

6 

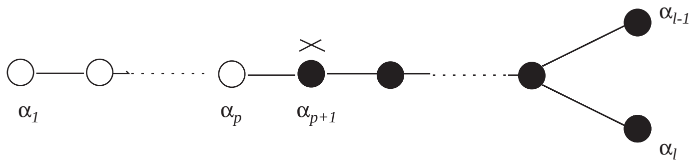
BI–BII

7 

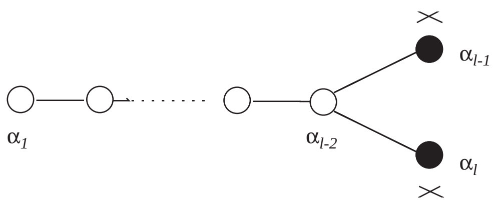
CII

8 

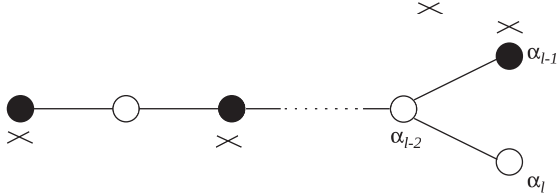
CII

9 

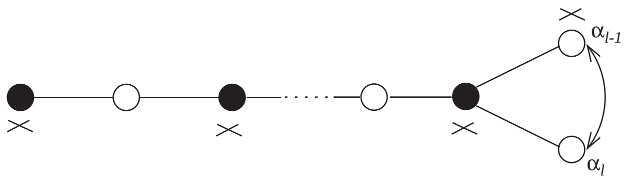
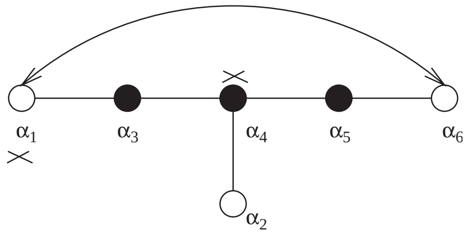
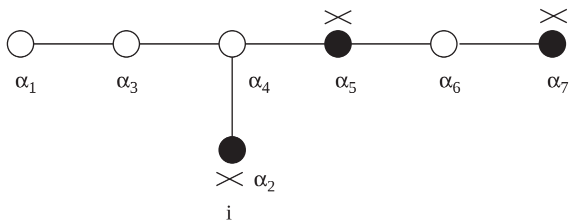
DI–DII

10 

DI

11 

DIII

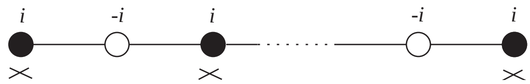
12		And the analogous with α_l crossed and α_{l-1} uncrossed	DIII
13		Or α_6 crossed α_1 uncrossed	EIII
14			EVI

To complete the proof we need to verify our claim. We do this by a direct inspection of the σ-diagrams in Table 3. Note that (D) is trivially true when $\#A = 1$.

Hence the proof will be divided in several cases.

Type A_l

AII: According to Table 3, the σ-diagram to discuss is the following:

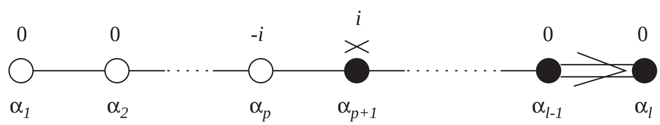


The numbers over the knots are the values assumed by λ on the fundamental roots. In this case property (D) holds because $\#A = 1$.

AIII-Subtype $\mathfrak{su}(p,q),\ p<q$ and AIV: In the σ-diagrams 2–5 in Table 3, for each fundamental root α we have $\lambda(\alpha) = -i$ if and only if $\alpha \in \Delta_\circ \backslash \Delta_\times$ and $\varepsilon_C(\alpha) \in \Delta_\times$. Assume A is connected, $A \cap \Delta_\times = \emptyset$ and $A \cap \lambda^{-1}(-i) \neq \emptyset$. In force of property (B), we have $\#\varepsilon_C(A \cap \Delta_\circ) \cap \Delta_\times = 1$, whence $\#A \cap \lambda^{-1}(-i) = 1$. This implies (D).

Type B_l

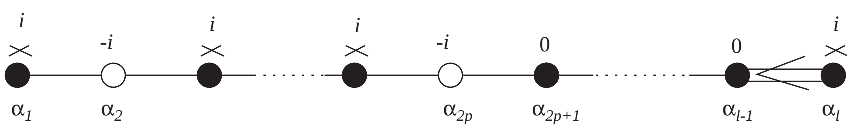
BI-Subtype $\mathfrak{so}(p,q),\ p<q$ and BII: The σ-diagram is



In this case A is of the form $A = \{\alpha_k \mid s \leqslant k \leqslant p\}$ for some $s \geqslant 1$ and $A \cap \lambda^{-1}(-i) = \{\alpha_p\}$. Since $\alpha_l \notin A$, the unique positive root with $\mathrm{supp}(\alpha) = A$ is $\sum_{\beta \in A} \beta$. Indeed, according to [4, Planche II, p. 252] we have that if $n_{\alpha,\alpha_j} = 2$, for some $j \geqslant 1$, then $n_{\alpha,\alpha_k} = 2$ for all $k \geqslant j$.

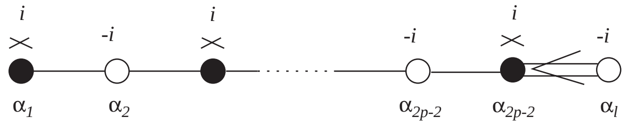
Type C_l

CII-Subtype $\mathfrak{sp}(p, l-p)$: The σ-diagrams to discuss are of the form



To verify (D), note that either $\#A = 1$ or $A = \{\alpha_k \mid 2p \leqslant k \leqslant s\}$ for some $s < l$, with $A \cap \lambda^{-1}(-i) = \{\alpha_{2p}\}$. It suffices to consider the second case. Assume $\mathrm{supp}(\alpha) = A$. Since $\alpha_l \in \mathrm{supp}(\alpha)$ if some $n_{\alpha,\alpha_j} \geqslant 2$ [4, Planche III, p. 254], we have $\alpha = \sum_{2p \leqslant k \leqslant s} \alpha_k$, proving (D).

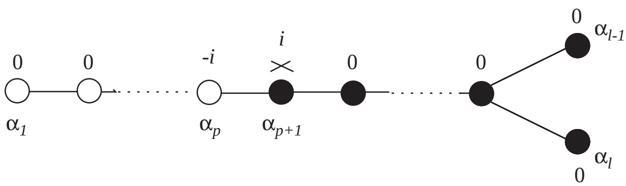
CII-Subtype $\mathfrak{sp}(p,p)$, $2p=l$: We have to consider the σ-diagram:



In this case the validity of (D) follows because $\#A=1$.

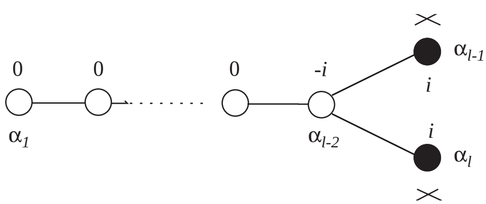
Type D_l

DI-Subtype $\mathfrak{so}(p,2l-p)$, $l \geqslant 4$, $1 \leqslant p < l-2$: The admissible σ-diagrams of this type are:



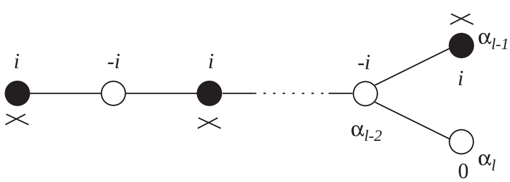
We have $A=\{\alpha_k \mid s \leqslant k \leqslant p\}$ with $s \geqslant 1$, and $A \cap \lambda^{-1}(-i)=\{\alpha_p\}$. To prove (D), it suffices to observe that $\alpha_l \in \mathrm{supp}(\alpha)$ if some $n_{\alpha,\alpha_j} \geqslant 2$.

DI-Subtype $\mathfrak{so}(l-2,l+2)$, $l \geqslant 4$: The σ-diagrams to consider are



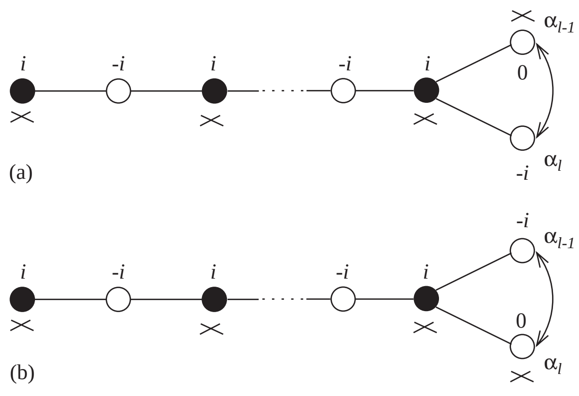
We have $A=\{\alpha_k \mid s \leqslant k \leqslant l-2\}$ for some $s \geqslant 1$ and $A \cap \lambda^{-1}(-i)=\{\alpha_{l-2}\}$. Hence as in the previous case we conclude that $\alpha=\sum_{s \leqslant k \leqslant l-2} \alpha_k$ and (D) follows.

DIII-Subtype $\mathfrak{u}^*_{2p}(\mathbb{H})$, $2p=l$: For the σ-diagram



we have either $\#A=1$ or $A=\{\alpha_{l-2},\alpha_l\}$. In this case the unique root with $\mathrm{supp}(\alpha)=A$ is $\alpha=\alpha_{l-2}+\alpha_l$ and this implies the validity of (D).

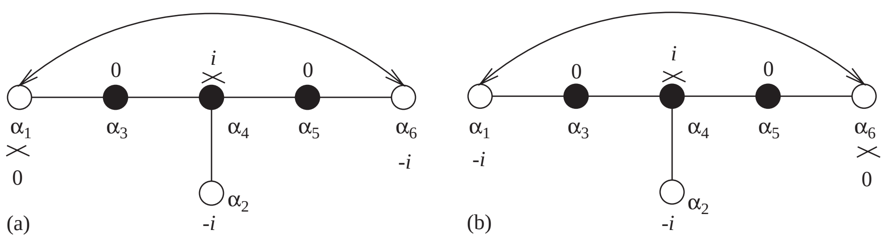
DIII-Subtype $\mathfrak{u}^*_{2p+1}(\mathbb{H})$, $2p+1=l$: From Table 3, we have the following admissible diagrams:



whence (D) is satisfied because $\#A = 1$.

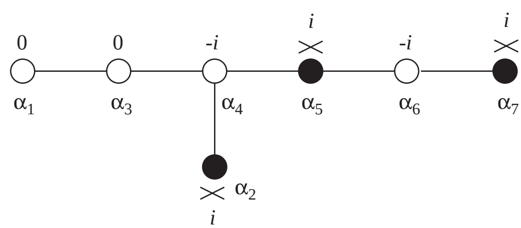
Exceptional types:

EIII: There are two σ-diagrams to examine



We observe that the only possibilities for A in diagram (a) are $A = \{\alpha_6\}$, $A = \{\alpha_2\}$ or $A = \{\alpha_5, \alpha_6\}$. Hence to prove (D), it suffices to remark that if a positive root α has a coefficient $n_{\alpha,\alpha_j} \geqslant 2$, then $\#\mathrm{supp}(\alpha) \geqslant 4$ [4, Planche V, p. 260]. A similar argument applies to diagram (b).

EVI: We have to examine the following σ-diagram:



We have either $\#A = 1$ or $A = \{\alpha_3, \alpha_4\}$ or $A = \{\alpha_1, \alpha_3, \alpha_4\}$. By using [4, Planche VI, p. 264], we see that there are no positive roots α with $\mathrm{supp}(\alpha) \subset \{\alpha_1, \alpha_3, \alpha_4\}$ and having a coefficient $n_{\alpha,\gamma} \geqslant 2$. This implies (D).

The proof is complete. $\square$

Remark 8.5. According to our classification, we have that, with the only exception of EIV, EVII and EIX, for each simple real Lie algebra $\mathfrak{g}$ whose Satake diagram contains black knotes, there exists a minimal parabolic CR algebra $(\mathfrak{g}, \mathfrak{q})$ having the (J) property and hence a nonstandard symmetric CR manifold. Taking account Table 2, and property (C) of our admissible σ-diagrams, the exceptions are due to the fact that the Satake diagram of EIV contains two elementary conjugation subdiagrams of type D_5.

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